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Is it Gaussian?
Testing bosonic quantum states

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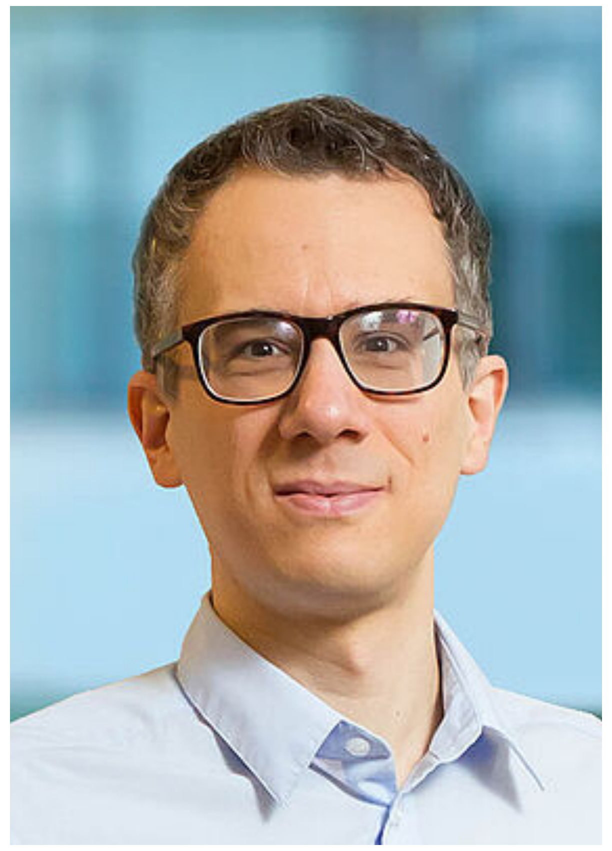


arXiv > quant-ph > arXiv:2510.07305

Setting and overview

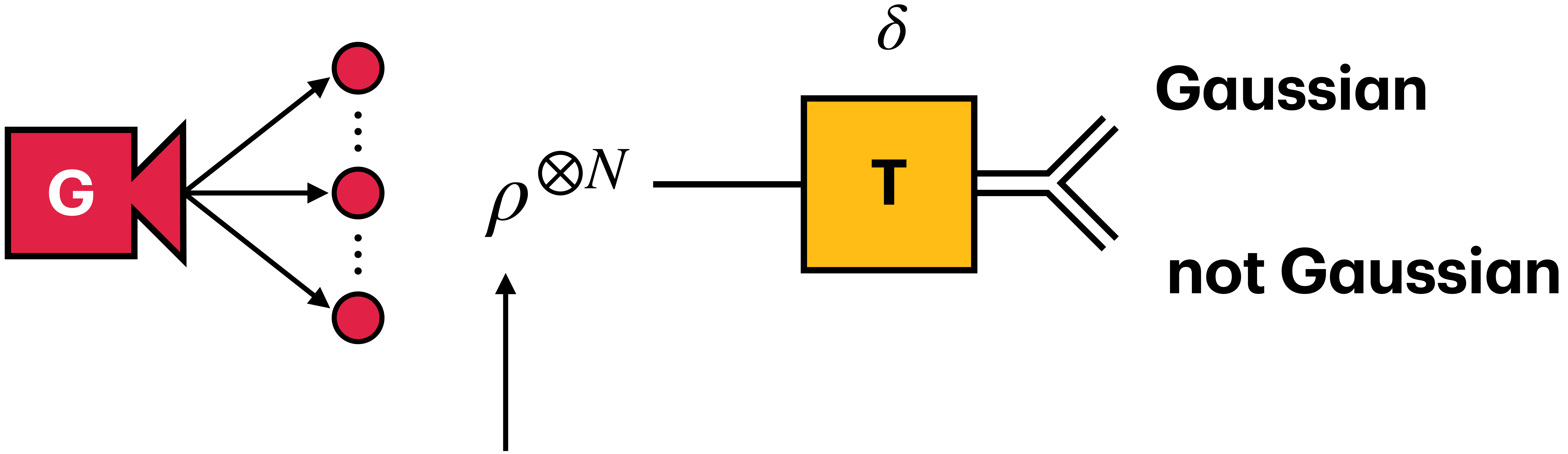
Quantum Physics

[Submitted on 8 Oct 2025]

Is it Gaussian? Testing bosonic quantum states

Filippo Girardi, Freek Witteveen, Francesco Anna Mele, Lennart Bittel, Salvatore F. E. Oliviero, David Gross, Michael Walter

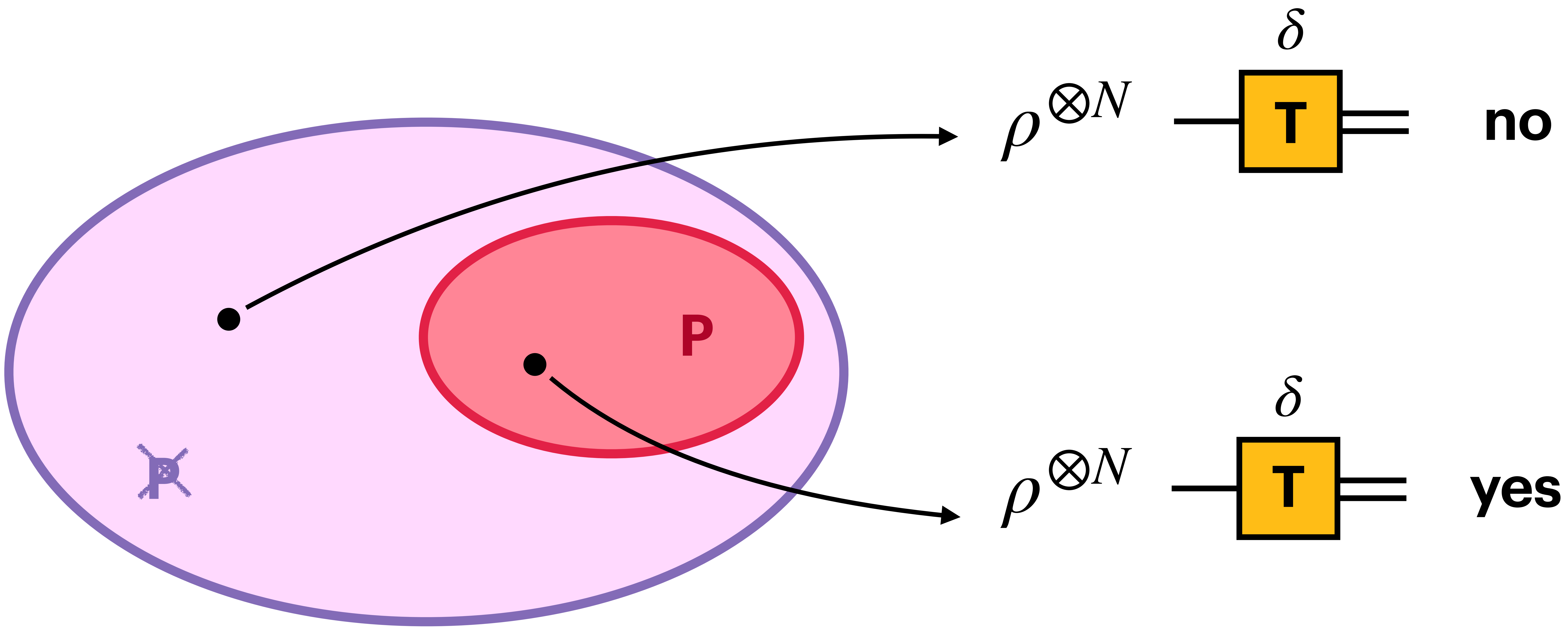
$\mathscr{H}_n = L^2(\mathbb{R}^n)$



assumptions $\Rightarrow$ how does N scale with the system size n?

Property testing

General setting



Property testing

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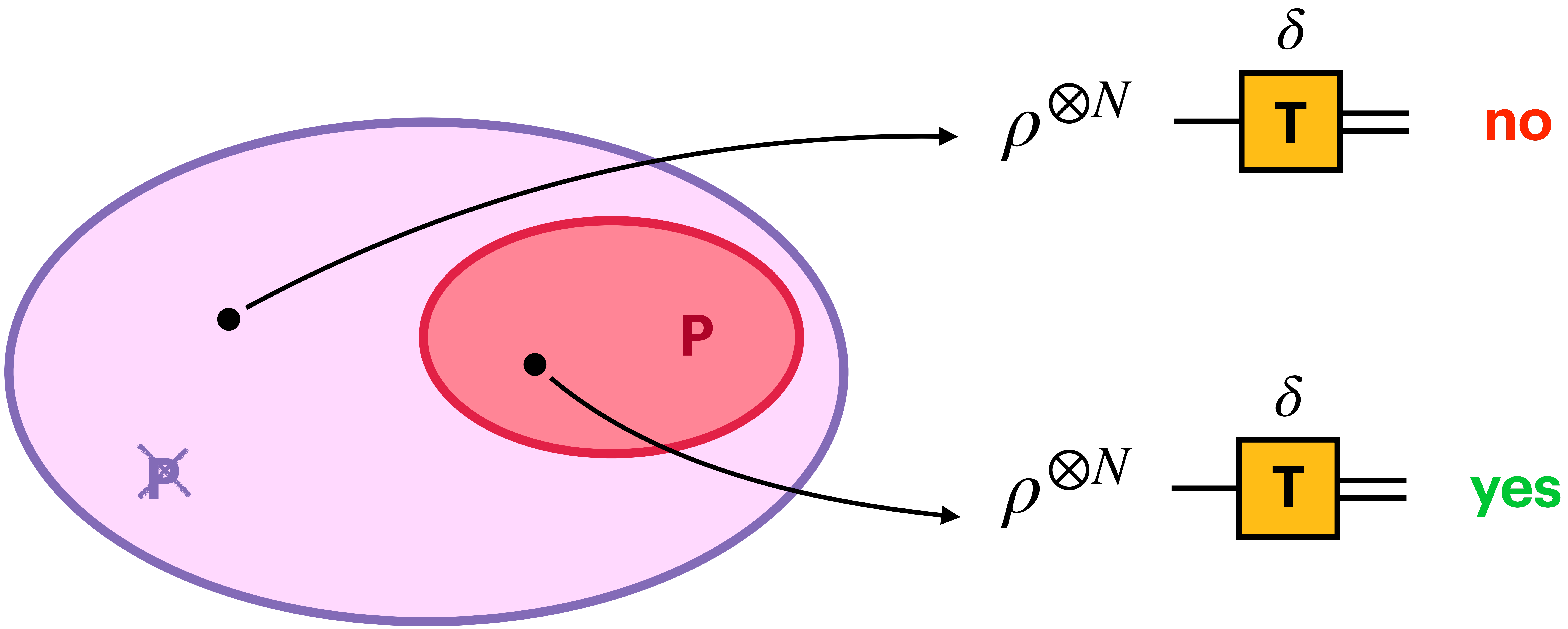
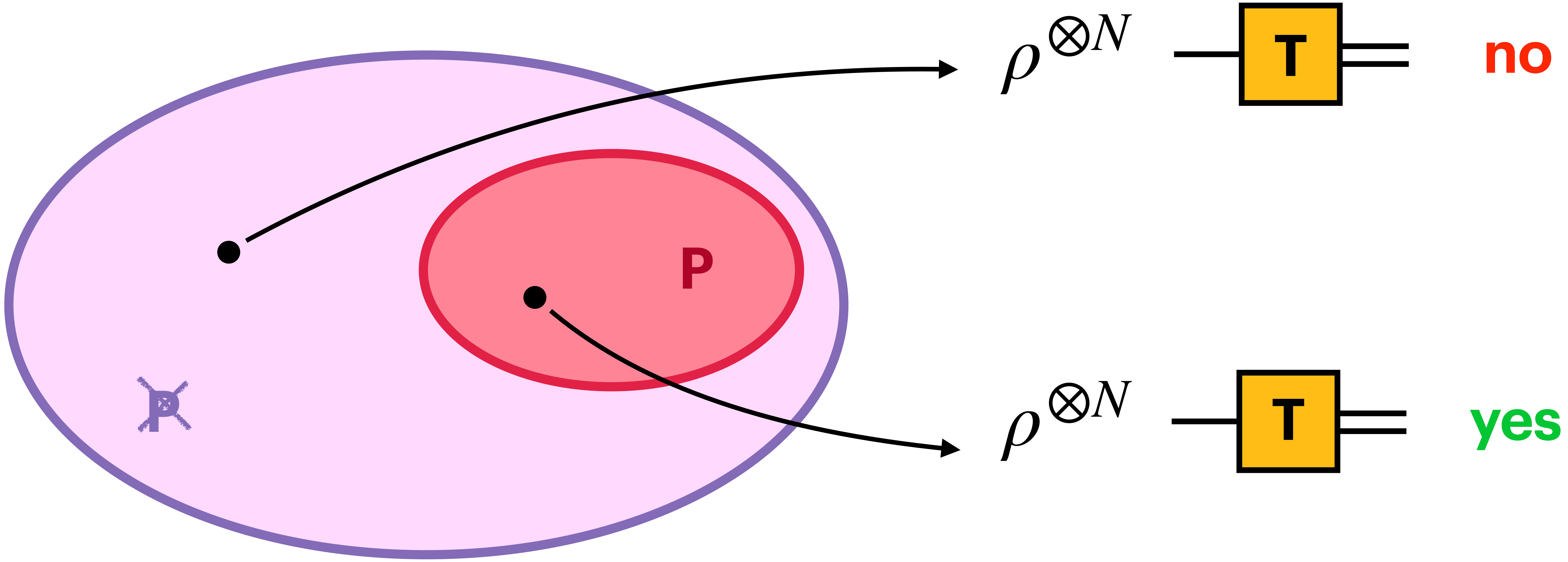


$$T \; : \; \{E, \mathbb{I} - E\}$$

$$0 \leq E \leq \mathbb{I}$$

Property testing

General setting

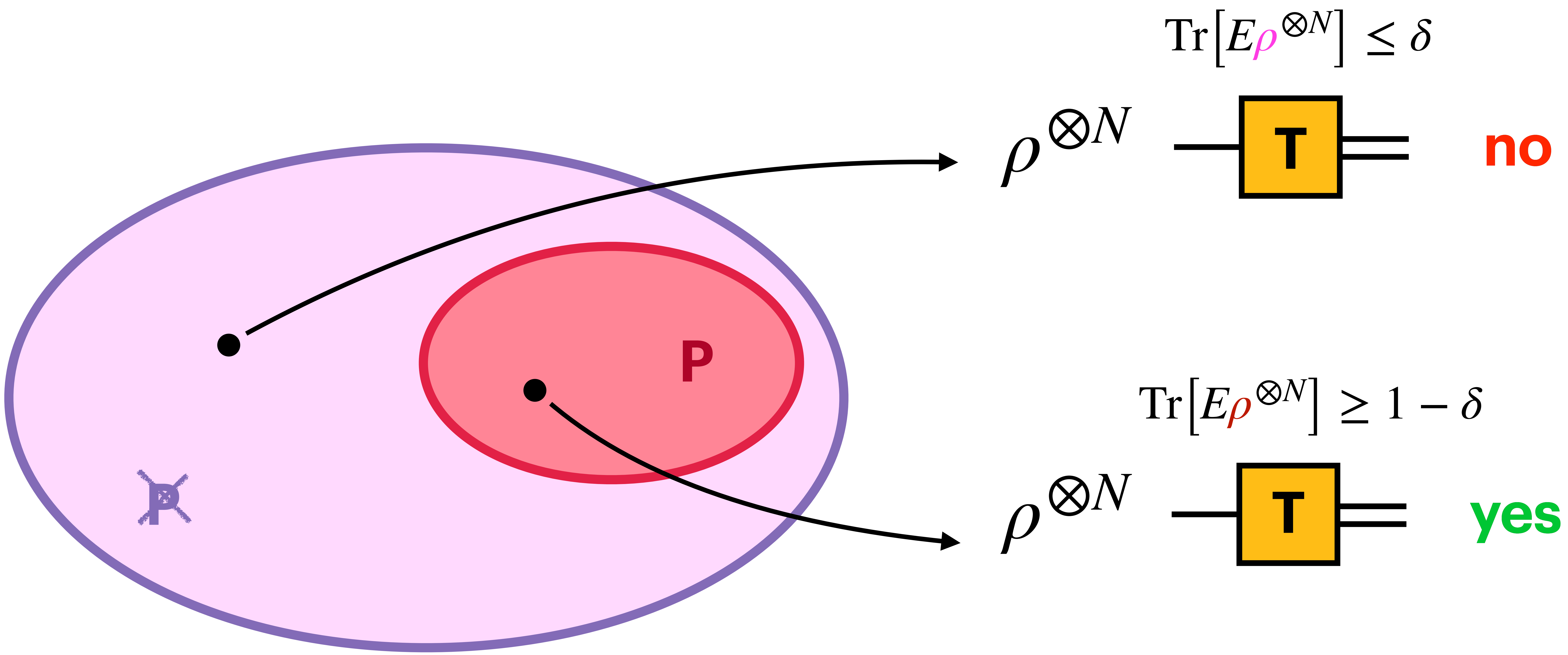


$$T \; : \; \{E, \mathbb{1} - E\}$$

$$0 \leq E \leq \mathbb{1}$$

Property testing

General setting

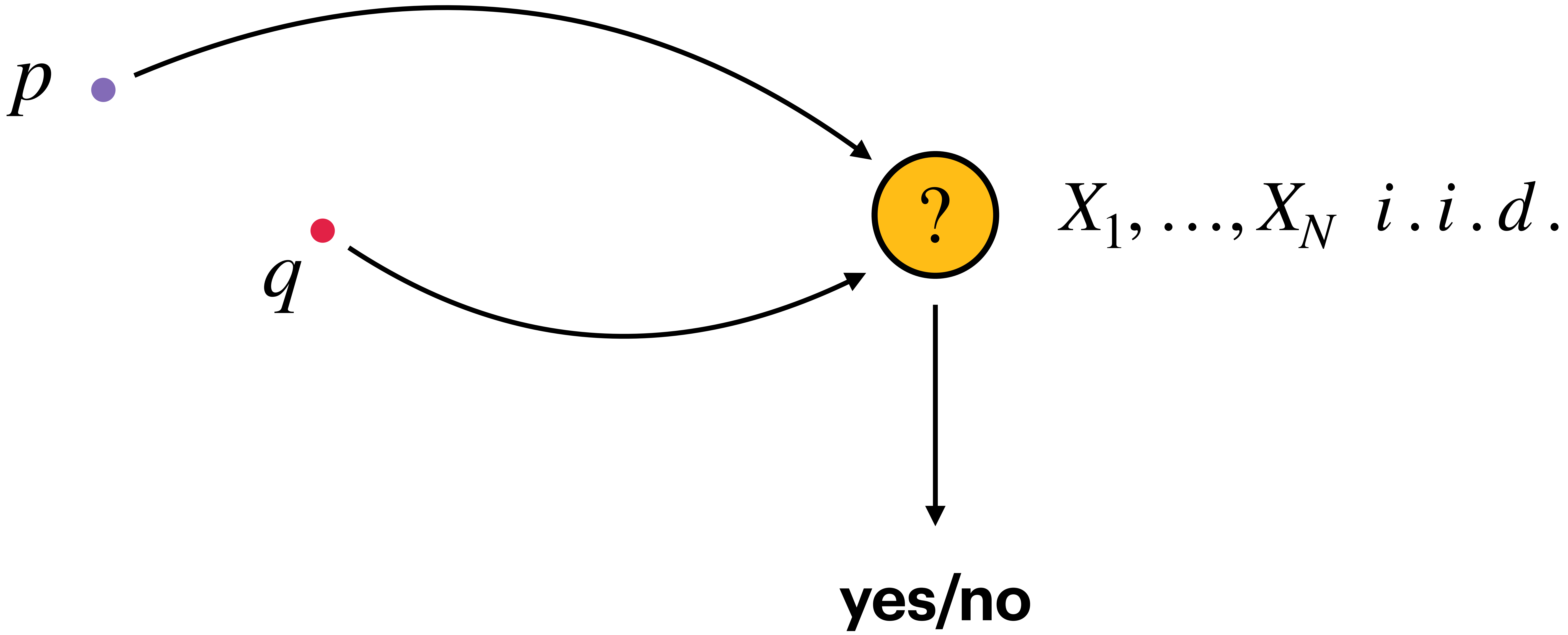


$$T \; : \; \{E, \mathbb{1} - E\}$$

$$0 \leq E \leq \mathbb{1}$$

Property testing

The simple case of identity testing



L. Paninski. IEEE Tr. Inf. Th. 54 (10), 4750-4755 ,
G&P. Valiant, FOCS, 2014,
C. Canonne, Found. Trends Commun. Inf. Theory, Vol. 19 No. 6 pp. 1032–1198

Property testing

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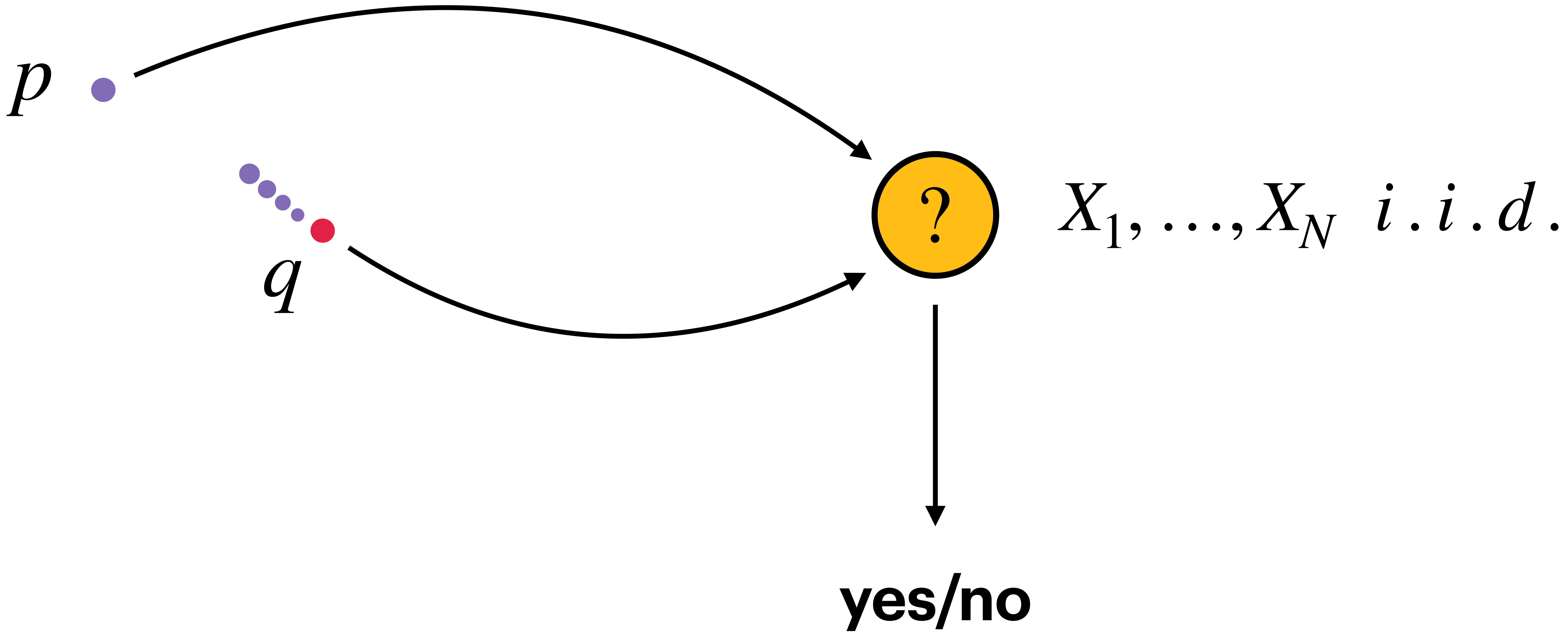

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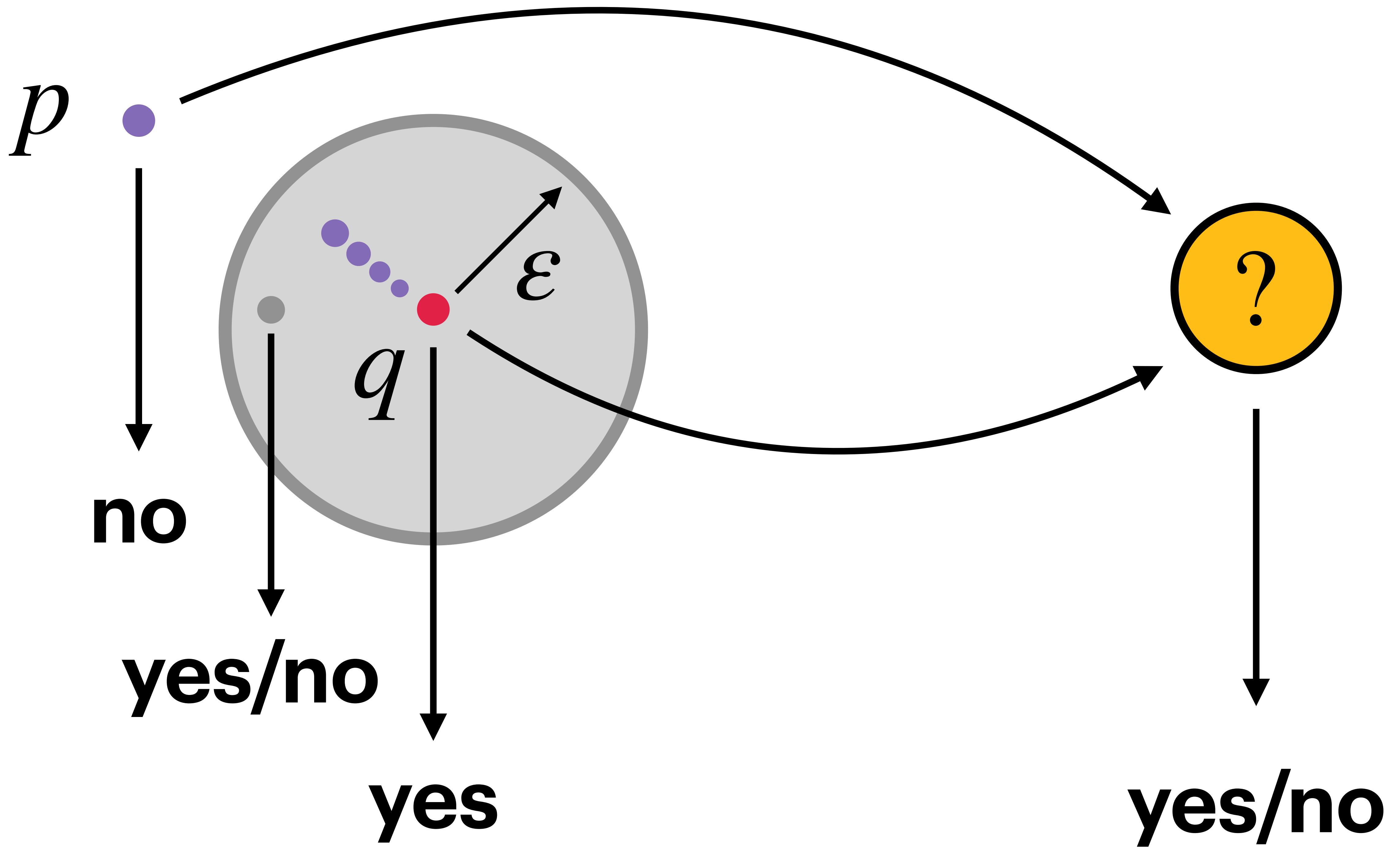

$X_1, \ldots, X_N \ i.i.d.$

Question: are we observing q

or any other p such that $\frac{1}{2}\|p - q\|_1 \geq \varepsilon$?

L. Paninski. IEEE Tr. Inf. Th. 54 (10), 4750-4755 ,
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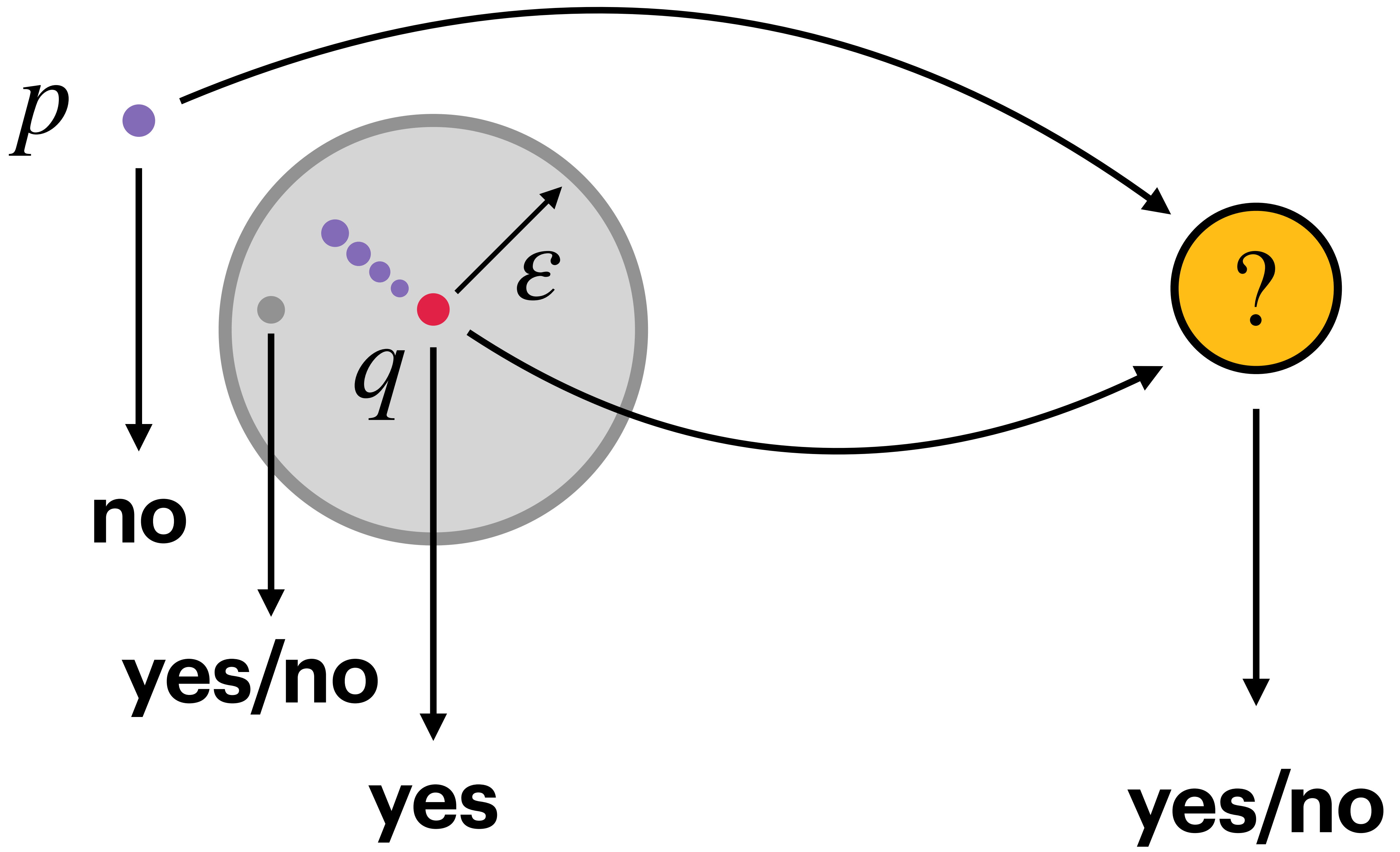

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$$\boxed{N = \Omega\left(\frac{1}{\varepsilon^2\|q\|_2}\right) \quad \left(\|q\|_\infty \leq \tfrac{1}{2}\right)}$$

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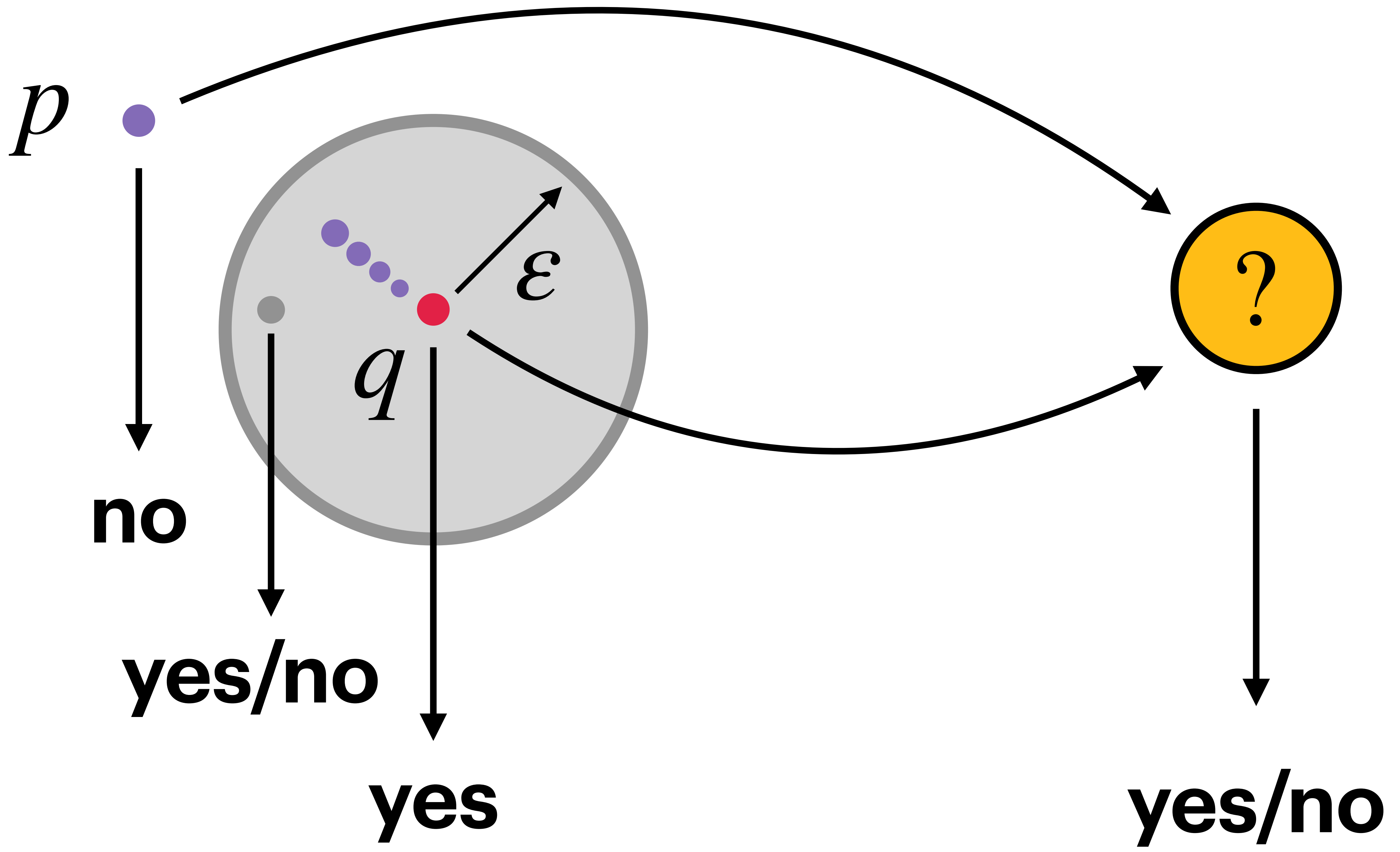

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$$\boxed{N = \Omega\left(\frac{\sqrt{|\mathcal{X}|}}{\varepsilon^2}\right) \quad q \text{ uniform}}$$

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Property testing

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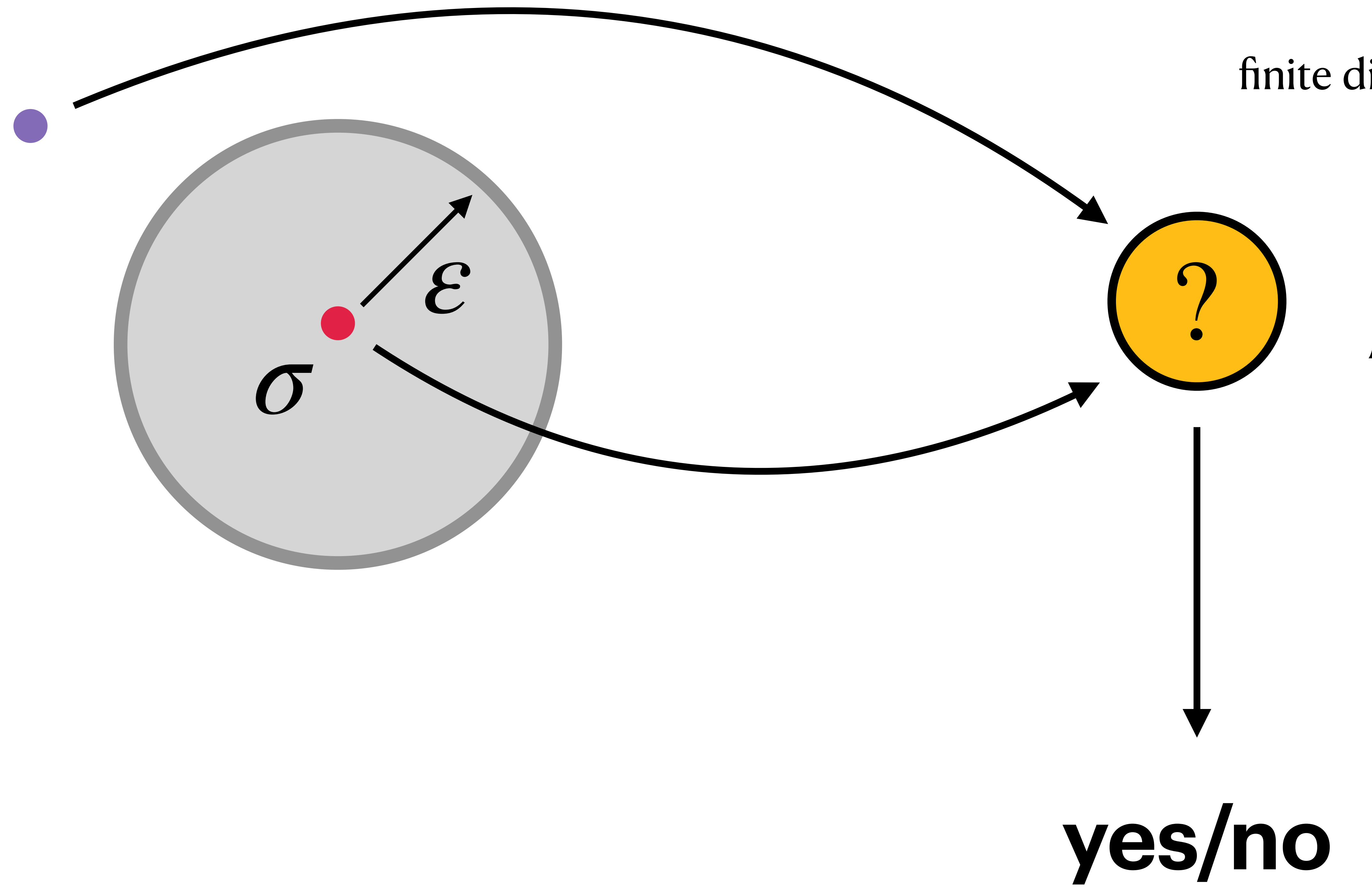


finite dimensional

$\rho^{\otimes N}$

Question: are we observing σ

or any other ρ such that $\dfrac{1}{2}\|\rho - \sigma\|_1 \geq \varepsilon$?

R. O'Donnell & J. Wright, Quantum Spectrum Testing, arXiv:1501.05028

R. O'Donnell & C. Wadhwa, Instance-Optimal Quantum State Certification with Entangled Measurements, arXiv:2507.06010

Gaussian states

Definition

Hilbert space: $\mathscr{H}_n = L^2(\mathbb{R}^n)$, where n is the number of modes (system size).

Quadrature operator vector: $\hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$.

Mean and covariance of a state: $\mathbf{m}(\rho) := \mathrm{Tr}[\hat{\mathbf{R}}\rho]$,

$$V(\rho) := \mathrm{Tr}[\{\hat{\mathbf{R}} - \mathbf{m}(\rho), (\hat{\mathbf{R}} - \mathbf{m}(\rho))^\intercal\}\rho] .$$

Energy of a state:

Gaussian state: $\rho = D_{\mathbf{m}} U_S \left(\bigotimes_{j=1}^{m} \frac{e^{-\xi_j a_j^\dagger a_j}}{\mathrm{Tr}\, e^{-\xi_j a_j^\dagger a_j}}, \right) U_S^\dagger D_{\mathbf{m}}^\dagger$

Gaussian states

Definition

A dangerously oversimplified analogy

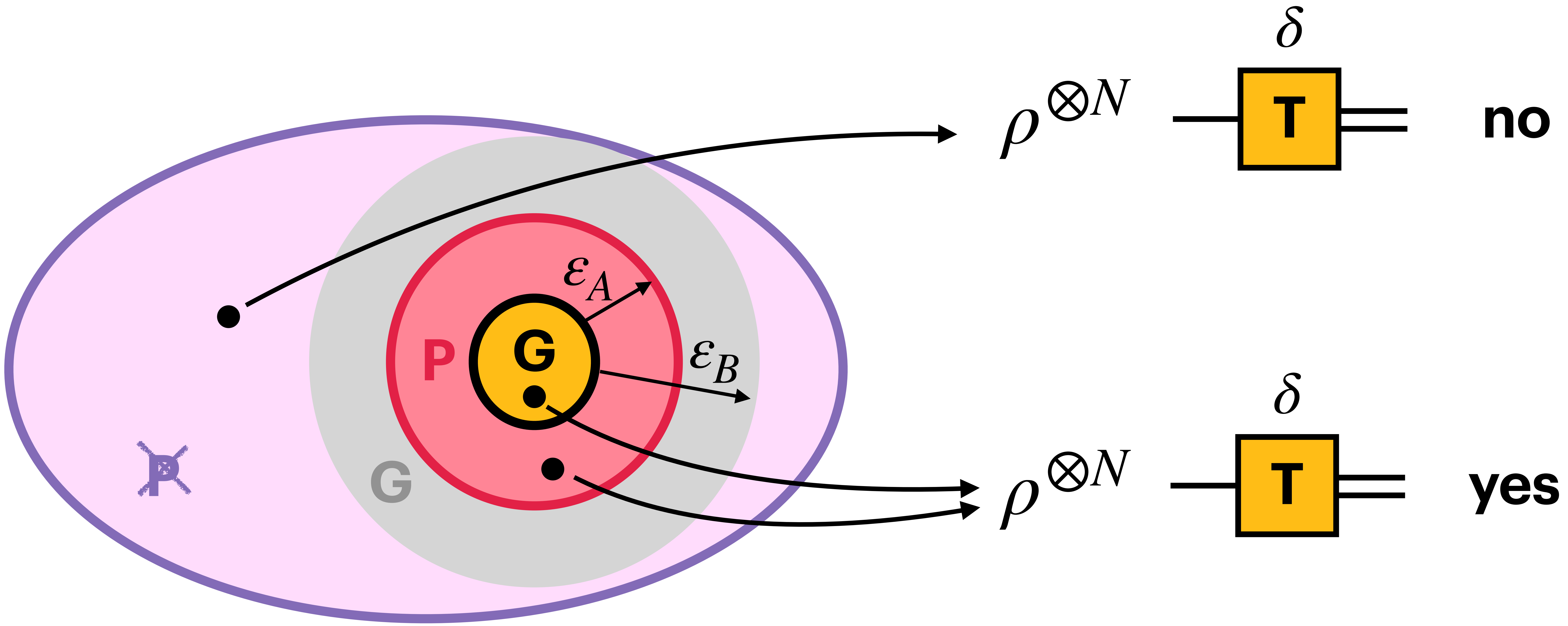
(with many caveats under the carpet)

probability distributions on an infinite set	probability distributions on $[d]$ with $d \to \infty$
continuous-variable (CV) systems	qudits with $d = +\infty$
Gaussian states	quantum generalisation of Gaussian distributions

Given a CV state, we can define its **energy**, its **mean** vector and its **covariance** matrix.
A Gaussian state is fully characterised by its mean vector and its covariance matrix.

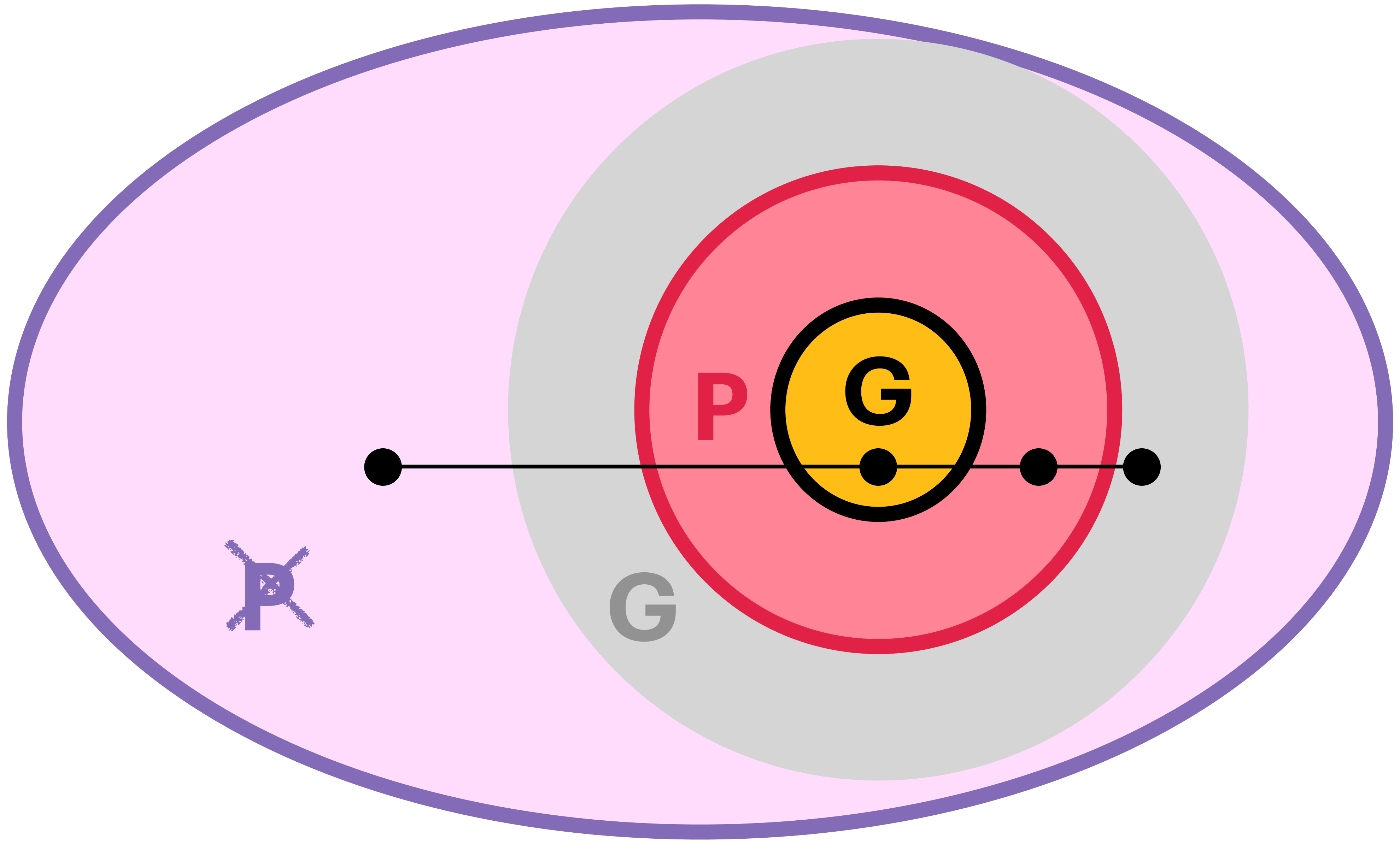
Gaussian property testing

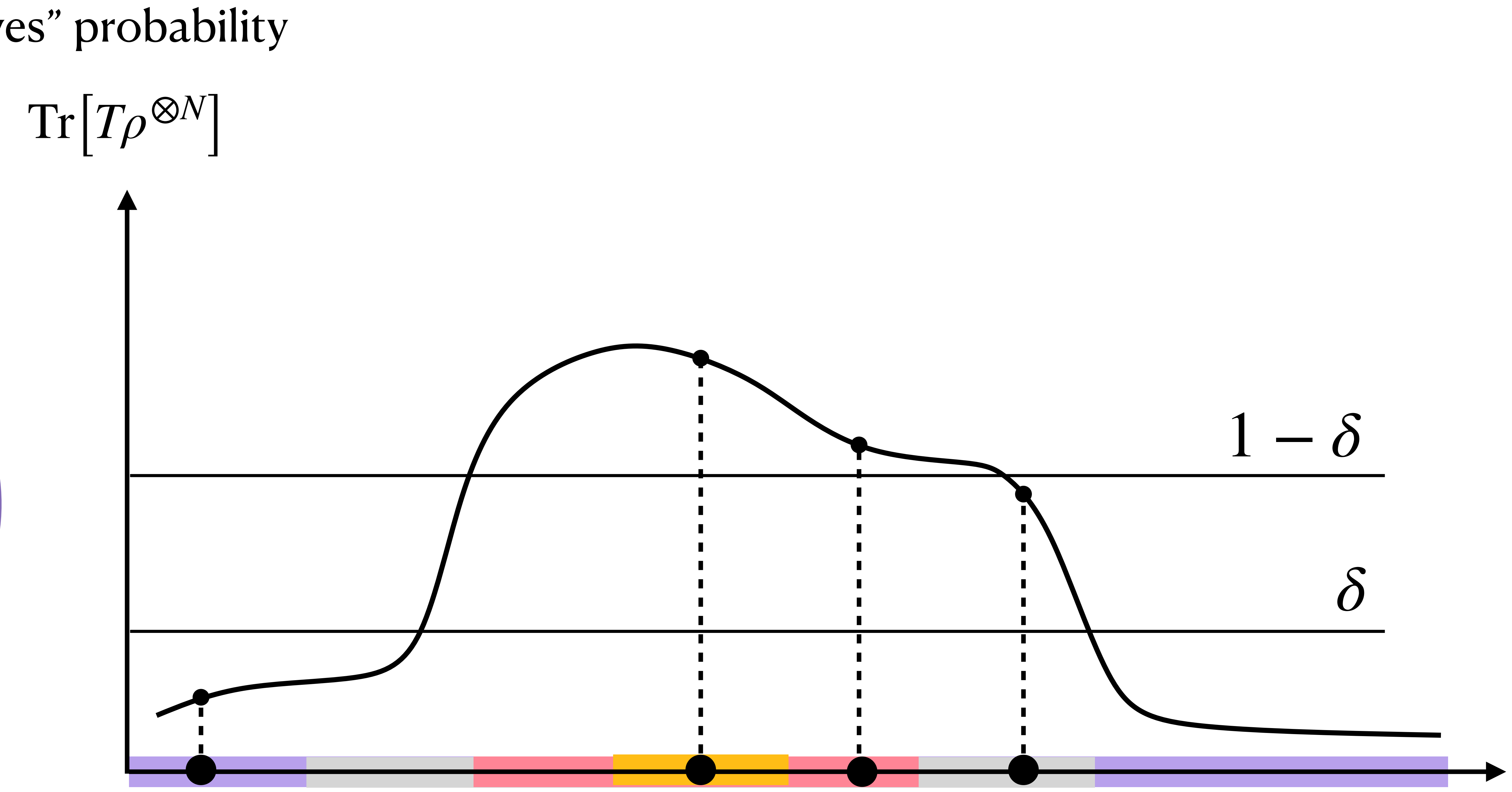
Tolerant testing



Gaussian property testing

Tolerant testing





Approach

- Study the properties of the subset to be tested (**symmetries**, characterisation)
- Choose the notion of distance: **trace distance** (but also relative entropy of non-Gaussianity)
- Technical tools: **trace distance bounds**
- Distinguish two regimes: testing “pure Gaussianity” and “mixed Gaussianity”

Pure Gaussian states

Definition of the problem

Let $0 \leq \varepsilon_A < \varepsilon_B$, let $0 < \delta \leq 1$ and let

- $\mathcal{G}_E$ be the set of **pure Gaussian states** with mean energy per mode at most E.

We say that an algorithm solves the property testing problem using N samples if, for any generic state ρ such that

A. either ρ is ε_A-close to $\mathcal{G}_E$, i.e. $\boxed{\min_{\psi \in \mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \varepsilon_A}$ **P**

B. or ρ is ε_B-far from $\mathcal{G}_E$, i.e. $\boxed{\min_{\psi \in \mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 > \varepsilon_B}$, ~~P~~

the algorithm can identify the underlying hypothesis A or B with failure probability at most δ.

Two approaches

symmetries
of Gaussian states

learning
Gaussian states

Example. Let $f \in C^2(\mathbb{R})$ such that $\boxed{U_{\pi/4}\, f \otimes f\, U^\dagger_{\pi/4} = f \otimes f}$, namely

$$f\left(\frac{x+y}{\sqrt{2}}\right) f\left(\frac{x-y}{\sqrt{2}}\right) = f(x) f(y) \qquad \forall\, x, y \in \mathbb{R}$$

Then

$$\log f\left(\frac{x+y}{\sqrt{2}}\right) + \log f\left(\frac{x-y}{\sqrt{2}}\right) = \log f(x) + \log f(y)$$

$$\Rightarrow \quad \partial_x \partial_y \log f\left(\frac{x+y}{\sqrt{2}}\right) = -\,\partial_x \partial_y \log f\left(\frac{x-y}{\sqrt{2}}\right) \quad \Rightarrow \quad \frac{d^2}{dt^2} \log f(t) = const\,. \quad \Rightarrow \quad \boxed{f \text{ is Gaussian.}}$$

See, e.g., E.H. Lieb, Gaussian kernels have only Gaussian maximizers. *Invent Math* **102**, 179–208 (1990).

Two approaches

symmetries
of Gaussian states

learning
Gaussian states

Proposition. Let U_θ be the rotation

$$(U_\theta f)(\mathbf{x},\mathbf{y}) = f(\cos\theta\,\mathbf{x} + \sin\theta\,\mathbf{y}, -\sin\theta\,\mathbf{x} + \cos\theta\,\mathbf{y}) \qquad \forall f \in L^2(\mathbb{R}^{2n}) \cong \mathscr{H}_n^{\otimes 2}.$$

For a pure state ψ on $\mathscr{H}_n$, the following properties are equivalent:

1. $U_\theta|\psi\rangle^{\otimes 2} = |\psi\rangle^{\otimes 2}$ for all $\theta \in [0,2\pi)$;
2. $U_\theta|\psi\rangle^{\otimes 2} = |\psi\rangle^{\otimes 2}$ is a product state for some θ which is not a multiple of $\pi/2$;
3. $|\psi\rangle$ is a pure Gaussian state with zero mean.

S. C. Springer & al. PRA 79, 062303 (2009)
J. Cuesta, J. Math. Phys. 61, 022201 (2020)

Two approaches

symmetries
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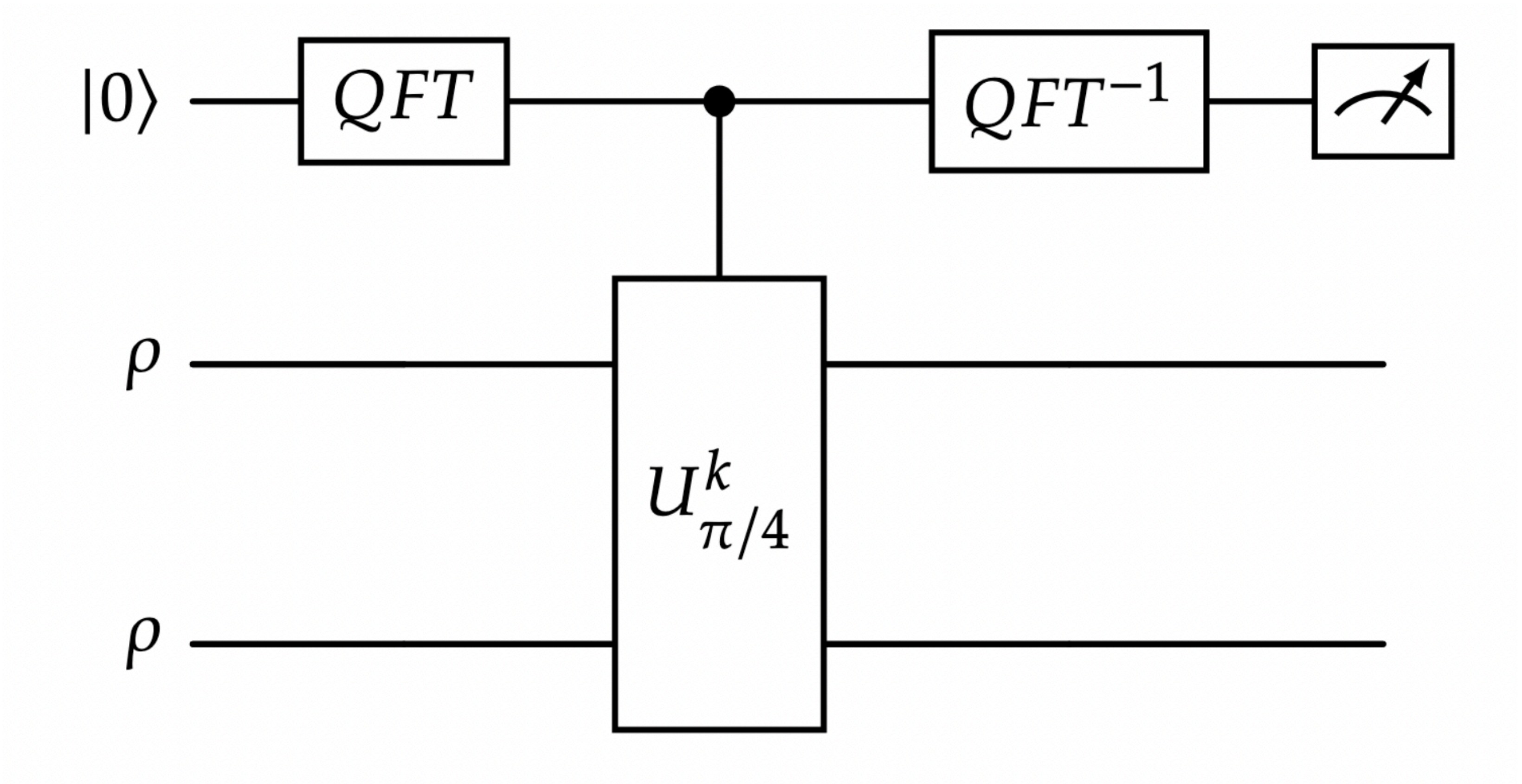
Gaussian states

Theorem. Take two copies of a quantum state ρ and measures whether $\rho^{\otimes 2}$ is in the rotation-invariant subspace. Then,

$$N = O(\varepsilon^{-2}\log(\delta^{-1})) \qquad \varepsilon := \Omega\left(\min\left(\varepsilon_B^2, \frac{1}{n^4E^4}\right) - \varepsilon_A\right)$$

copies of ρ are sufficient to test its closeness to the set of zero-mean pure Gaussian states.

Implementation. $\dim \mathscr{H}_A = 8$



Generalisations. Non-zero mean, smaller auxiliary systems, testing the generator of rotations $\langle G^2\rangle_{\rho^{\otimes 2}}$.

Two approaches

symmetries
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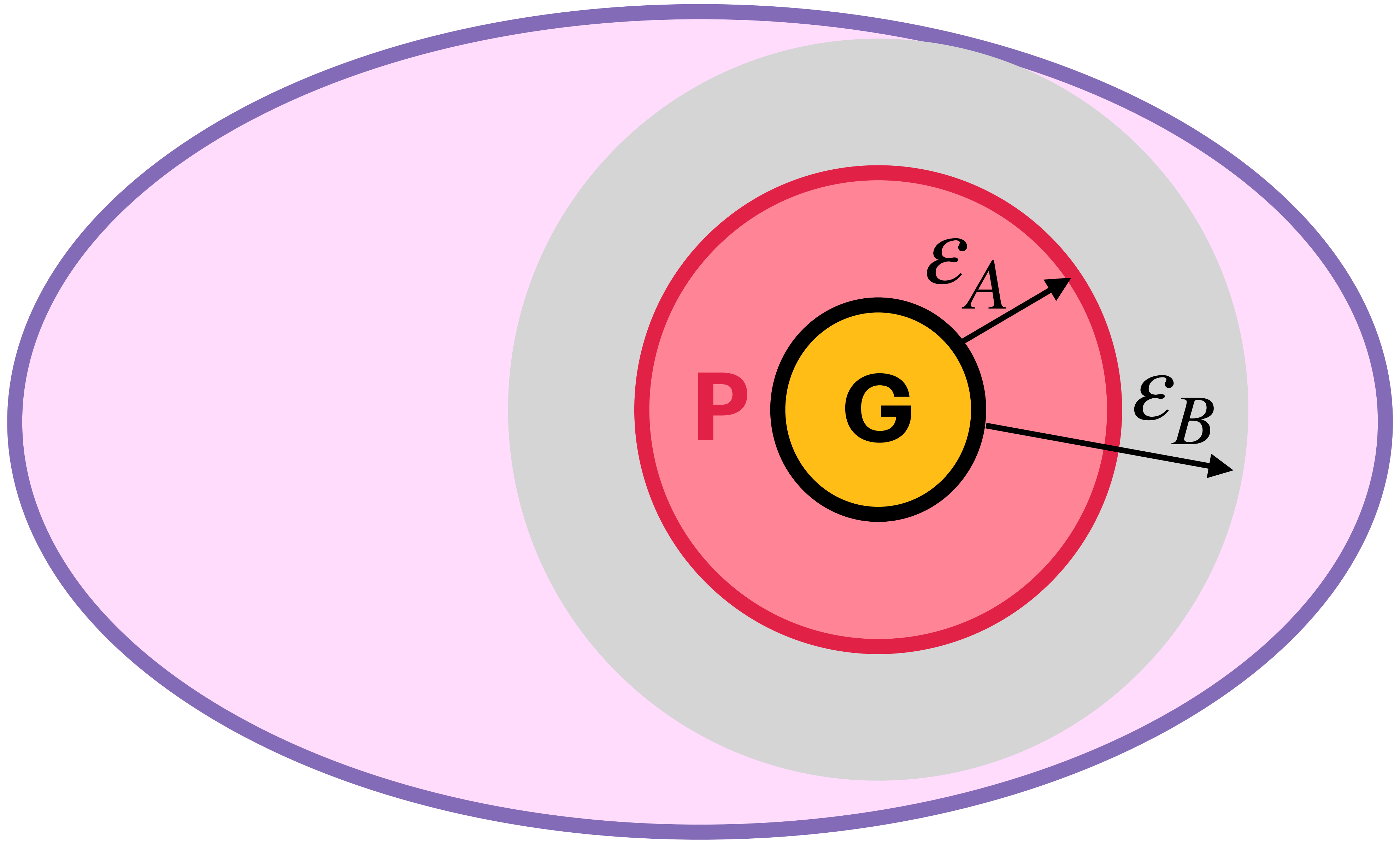
learning
Gaussian states

Idea. ρ is a pure Gaussian state iff its symplectic eigenvalues $\{\nu_i\}$ are all equal to 1.

$V = SDS^\intercal$

Sketch of the algorithm.



For the fermonic case,
see Bittel & al., PRX Quantum 6, 030341 (2025)

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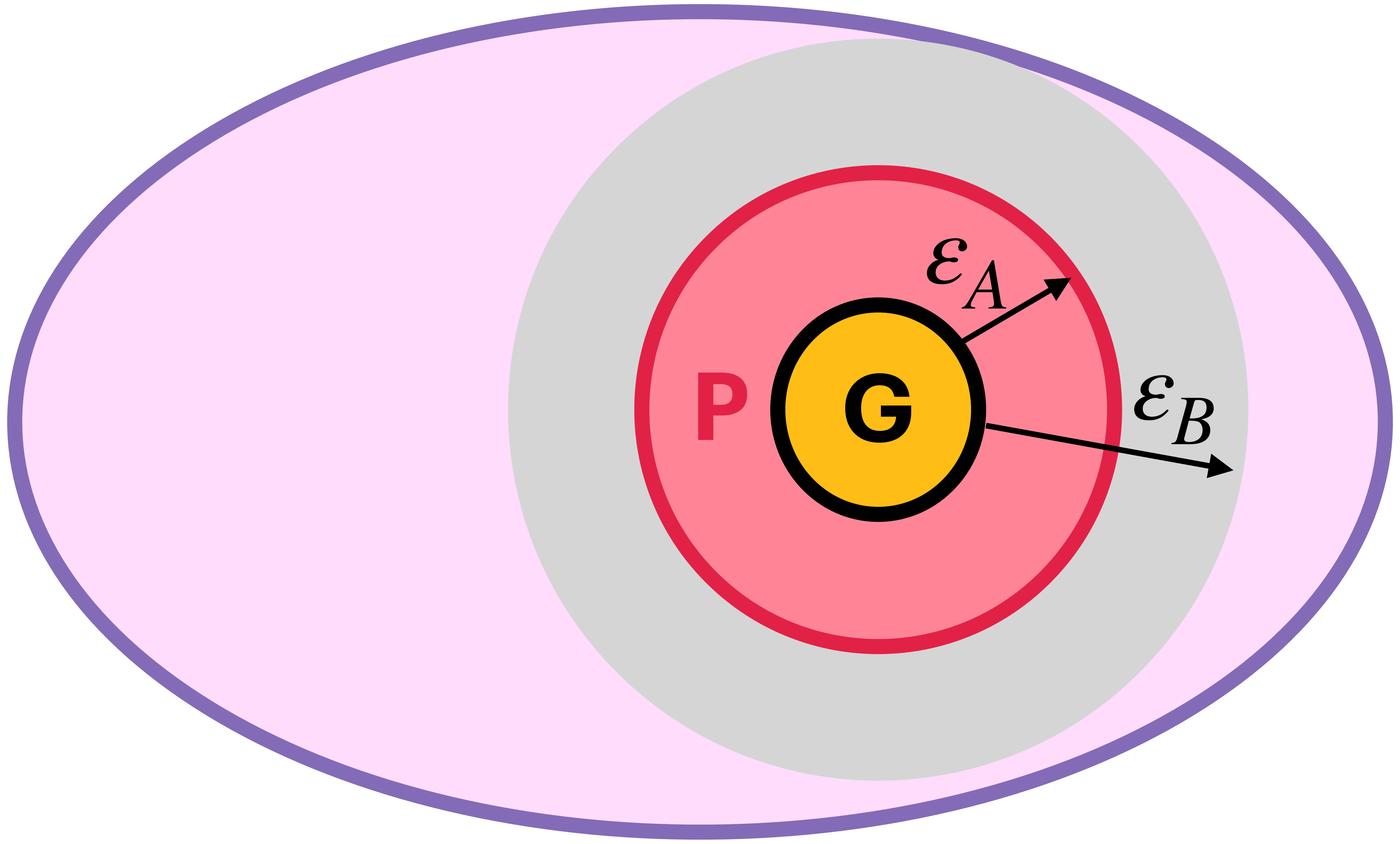
$\rho^{\otimes N} \longrightarrow$  $= \tilde{V} = \tilde{\nu}_{\max}$ vs ν_{thr}



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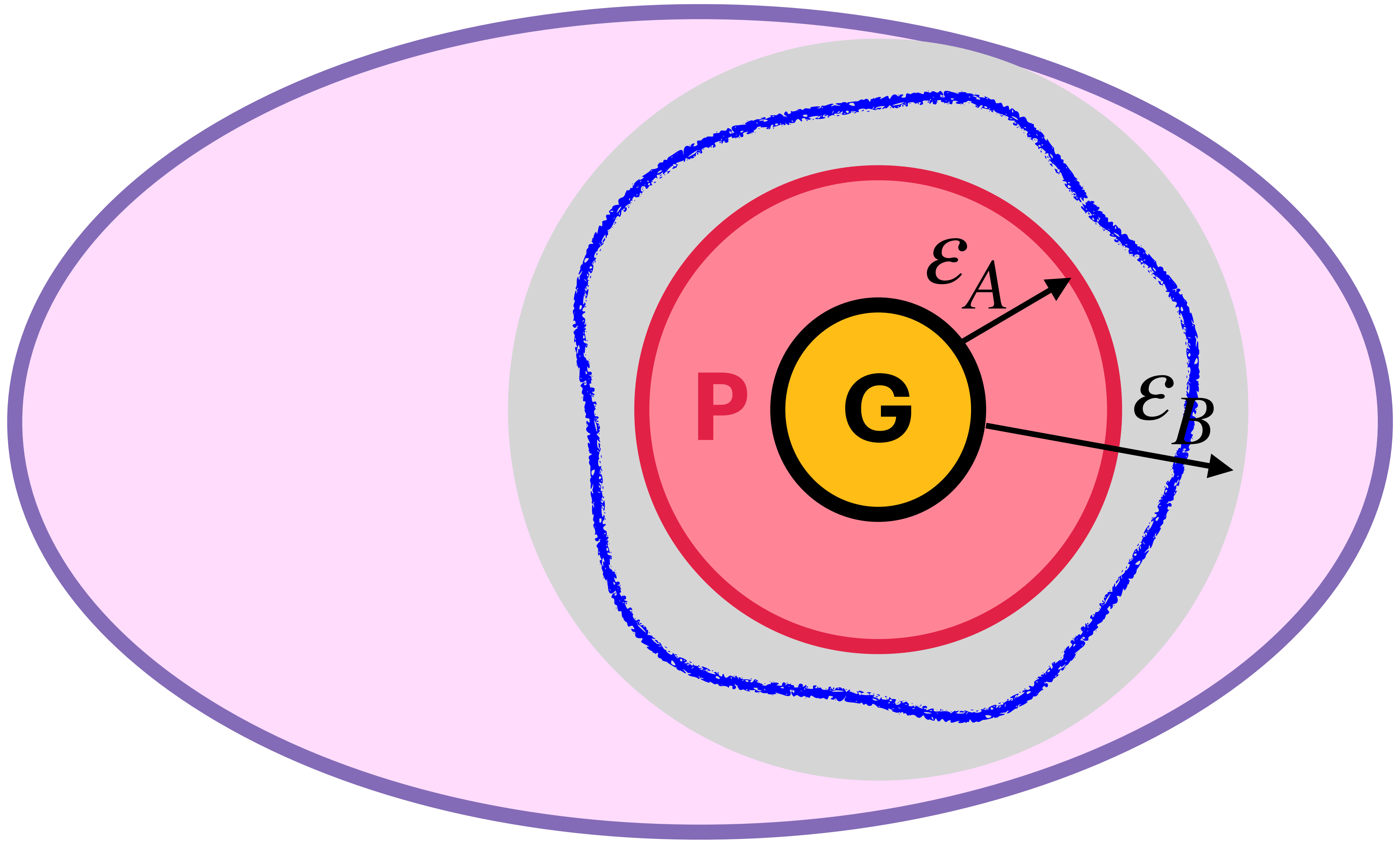
Sketch of the algorithm.

$\rho^{\otimes N} \longrightarrow$  $=\!=\; \tilde{V} \;=\!=\; \tilde{\nu}_{\max}$ vs ν_{thr}

$\nu_{\text{thr}} = 1 + f(\varepsilon_A, \varepsilon_B, n, E)$



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Two approaches

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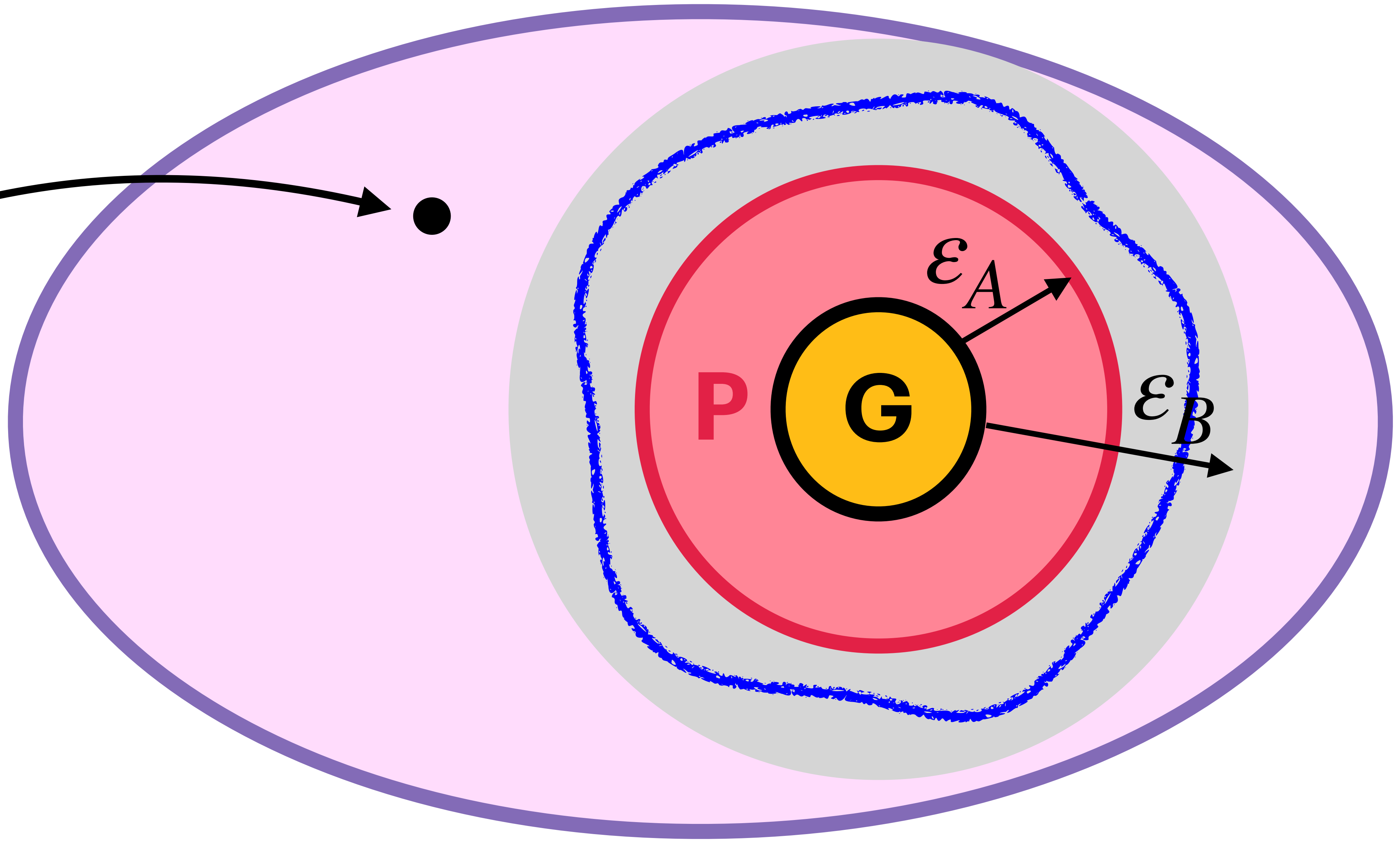

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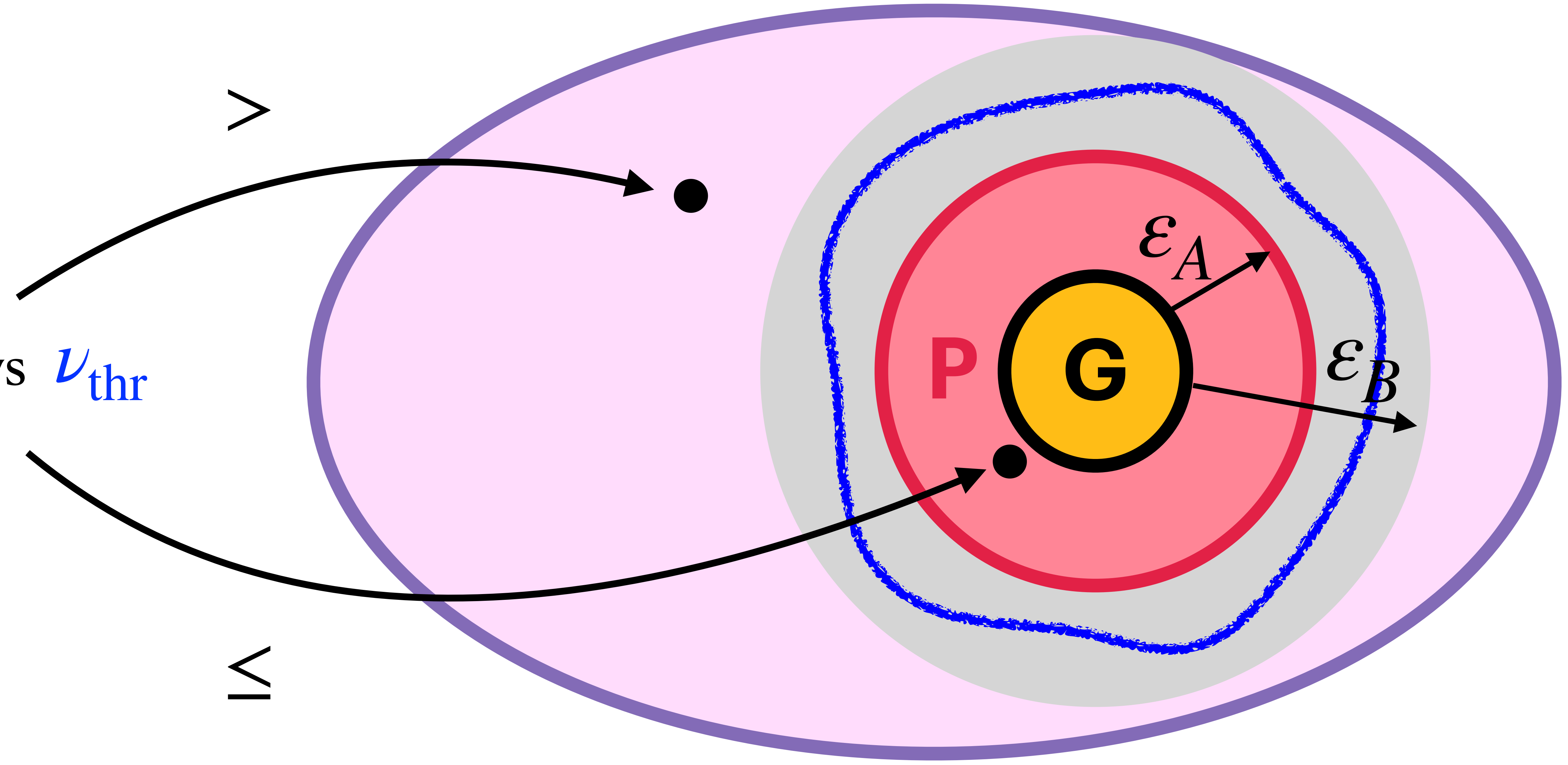

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For the fermonic case,
see Bittel & al., PRX Quantum 6, 030341 (2025)

Learning the covariance matrix

Lemma 41 (Sample complexity of estimating the covariance matrix). *Let $\varepsilon, \delta \in (0,1)$ and $E > 0$. Let ρ be an n-mode quantum state satisfying with second moment of the energy upper bounded by nE, i.e. $\sqrt{\mathrm{Tr}\left[\hat{E}^2\rho\right]} \leq nE$. Then, a number*

$$(n+3)\left\lceil 68\log\left(\frac{2(2n^2+3n)}{\delta}\right)\frac{200(8n^2E^2+3n)}{\varepsilon^2}\right\rceil = \mathcal{O}\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon^2}\right), \tag{C30}$$

of copies of ρ are sufficient to build a vector $\tilde{\mathbf{m}} \in \mathbb{R}^{2n}$ and a symmetric matrix $\tilde{V}' \in \mathbb{R}^{2n,2n}$ such that

$$\Pr\left(\|\tilde{V}' - V(\rho)\|_2 \leq \varepsilon \quad \textit{and} \quad \tilde{V}' + i\Omega \geq 0 \quad \textit{and} \quad \|\tilde{\mathbf{m}} - \mathbf{m}(\rho)\| \leq \frac{\varepsilon}{10\sqrt{8nE}}\right) \geq 1-\delta\,. \tag{C31}$$

Such procedure only requires single-copy measurements.

F. A. Mele, A. A. Mele, L. Bittel, J. Eisert, V. Giovannetti, L. Lami, L. Leone, S. F. E. Oliviero,
Learning quantum states of continuous variable systems, Nature Physics 21, 2002-2008 (2025)

Trace distance bounds

Lemma 10 (Upper bound on the trace distance between Gaussian states [4]). *The trace distance between two Gaussian states $\rho(V,\mathbf{m})$ and $\rho(W,\mathbf{t})$ can be upper bounded as follows:*

$$\frac{1}{2}\|\rho(V,\mathbf{m})-\rho(W,\mathbf{t})\|_1 \le \frac{1+\sqrt{3}}{8}\max(\operatorname{Tr}V,\operatorname{Tr}W)\,\|V-W\|_\infty + \sqrt{\frac{\min(\|V\|_\infty,\|W\|_\infty)}{2}}\|\mathbf{m}-\mathbf{t}\|\,. \tag{A19}$$

Lemma 11 (Upper bound on the trace distance between an arbitrary state and a pure Gaussian state [3]). *Let ψ_G be a pure Gaussian state with covariance matrix V and first moment $\mathbf{m}$. Moreover, let σ be a (possibly non-Gaussian and possibly mixed) state with covariance matrix W and first moment $\mathbf{t}$. Then, it holds that*

$$\frac{1}{2}\|\psi_G-\sigma\|_1 \le \sqrt{E}\sqrt{\|V-W\|_\infty + 2\|\mathbf{m}-\mathbf{t}\|^2}\,, \tag{A20}$$

where $E := \max(\operatorname{Tr}[\sigma\hat{E}],\operatorname{Tr}[\psi_G\hat{E}])$ is the maximum energy and $\hat{E}$ denotes the energy operator.

L. Bittel, F. A. Mele, A. A. Mele, S. Tirone, L. Lami
Optimal estimates of trace distance between bosonic Gaussian states and applications to learning, Quantum 9, 1769 (2025)

F. A. Mele, A. A. Mele, L. Bittel, J. Eisert, V. Giovannetti, L. Lami, L. Leone, S. F. E. Oliviero,
Learning quantum states of continuous variable systems, Nature Physics 21, 2002-2008 (2025)

Trace distance bounds

Lemma 12 (Lower bound on the trace distance between Gaussian states [3]). *The trace distance between two Gaussian states $\rho(V,\mathbf{m})$ and $\rho(W,\mathbf{t})$ can be lower bounded in terms of the norm distance between their first moments and the norm distance between their covariance matrices as*

$$\begin{aligned}\frac{1}{2}\|\rho(V,\mathbf{m})-\rho(W,\mathbf{t})\|_1 &\geq \frac{1}{200}\min\left\{1,\frac{\|\mathbf{m}-\mathbf{t}\|}{\sqrt{4\min(\|V\|_\infty,\|W\|_\infty)+1}}\right\}, \\ \frac{1}{2}\|\rho(V,\mathbf{m})-\rho(W,\mathbf{t})\|_1 &\geq \frac{1}{200}\min\left\{1,\frac{\|V-W\|_2}{4\min(\|V\|_\infty,\|W\|_\infty)+1}\right\}.\end{aligned} \tag{A21}$$

Lemma 13 (Lower bound on the trace distance between arbitrary states [18]). *Let ρ be a (possibly non-Gaussian) state with first moment $\mathbf{m}$ and covariance matrix V. Moreover, let σ be a (possibly non-Gaussian) state with first moment $\mathbf{t}$ and covariance matrix W. The trace distance can be lower bounded as*

$$\frac{1}{2}\|\rho-\sigma\|_1 \geq \frac{\|\mathbf{m}-\mathbf{t}\|^2}{32\max\left(\mathrm{Tr}\left[\hat{E}\rho\right],\mathrm{Tr}\left[\hat{E}\sigma\right]\right)}, \tag{A22}$$

$$\frac{1}{2}\|\rho-\sigma\|_1 \geq \frac{\|V-W\|_\infty^2}{3098\max(\mathrm{Tr}[\hat{E}^2\rho],\mathrm{Tr}[\hat{E}^2\sigma])}, \tag{A23}$$

where $\hat{E}$ denotes the energy operator.

F. A. Mele, A. A. Mele, L. Bittel, J. Eisert, V. Giovannetti, L. Lami, L. Leone, S. F. E. Oliviero, *Learning quantum states of continuous variable systems,* Nature Physics 21, 2002-2008 (2025)

F. A. Mele, S. F. E. Oliviero, V. Upreti, U. Chabaud *The symplectic rank of non-Gaussian quantum states,* arXiv:2504.19319 [quant-ph]

Perturbation bounds

Lemma 36 (Perturbation on symplectic diagonalisation [71]). *Let $V_1, V_2 \in \mathbb{R}^{2n\times 2n}$ be two covariance matrices with symplectic diagonalisations $V_1 = S_1 D_1 S_1^\intercal$ and $V_2 = S_2 D_2 S_2^\intercal$, where the elements on the diagonal of D_1 and D_2 are arranged in descending order. Then*

$$\begin{aligned} \|D_1 - D_2\|_\infty &\leq \sqrt{K(V_1)K(V_2)}\|V_1 - V_2\|_\infty , \\ \|D_1 - D_2\|_2 &\leq \sqrt{K(V_1)K(V_2)}\|V_1 - V_2\|_2 , \end{aligned} \tag{C5}$$

where $K(V)$ is the condition number of the covariance matrix V, defined as $K(V) := \|V\|_\infty \|V^{-1}\|_\infty$.

M. Idel, D. Lercher, M. M. Wolf, *An operational measure for squeezing,* J. Phys. A: Math. Theor. 49 445304 (2016)

Recap

$$\rho^{\otimes N} \longrightarrow \quad = \tilde{V} = \tilde{\nu}_{\max} \text{ vs } \nu_{\text{thr}}$$

Learning mean and covariance.

$$N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon^2}\right)$$
$$\mathbb{P}\left(\|\tilde{V}' - V(\rho)\|_2 \le \varepsilon,\ \tilde{V}' + i\Omega \ge 0,\ \|\tilde{\mathbf{m}} - \mathbf{m}(\rho)\| \le \frac{\varepsilon}{10\sqrt{8nE}}\right) \ge 1 - \delta$$

Perturbation on symplectic diagonalisation.

$$\|D_1 - D_2\|_\infty \le \sqrt{K(V_1)K(V_2)}\,\|V_1 - V_2\|_\infty$$
$$\|D_1 - D_2\|_2 \le \sqrt{K(V_1)K(V_2)}\,\|V_1 - V_2\|_2$$

Upper bound between Gaussian states.

$$\frac{1}{2}\|\rho(V,\mathbf{m}) - \rho(W,\mathbf{t})\|_1 \le \frac{1+\sqrt{3}}{8}\max(\mathrm{Tr}V, \mathrm{Tr}W)\,\|V - W\|_\infty + \sqrt{\frac{\min(\|V\|_\infty, \|W\|_\infty)}{2}}\|\mathbf{m} - \mathbf{t}\|$$

Upper bound between a pure Gaussian state and a generic state.

$$\frac{1}{2}\|\psi_G - \sigma\|_1 \le \sqrt{E}\sqrt{\|V - W\|_\infty + 2\|\mathbf{m} - \mathbf{t}\|^2}$$

Lower bound between Gaussian states.

$$\frac{1}{2}\|\rho(V,\mathbf{m}) - \rho(W,\mathbf{t})\|_1 \ge \frac{1}{200}\min\left\{1, \frac{\|\mathbf{m} - \mathbf{t}\|}{\sqrt{4\min(\|V\|_\infty, \|W\|_\infty) + 1}}\right\}$$
$$\frac{1}{2}\|\rho(V,\mathbf{m}) - \rho(W,\mathbf{t})\|_1 \ge \frac{1}{200}\min\left\{1, \frac{\|V - W\|_2}{4\min(\|V\|_\infty, \|W\|_\infty) + 1}\right\}$$

Lower bound between arbitrary states.

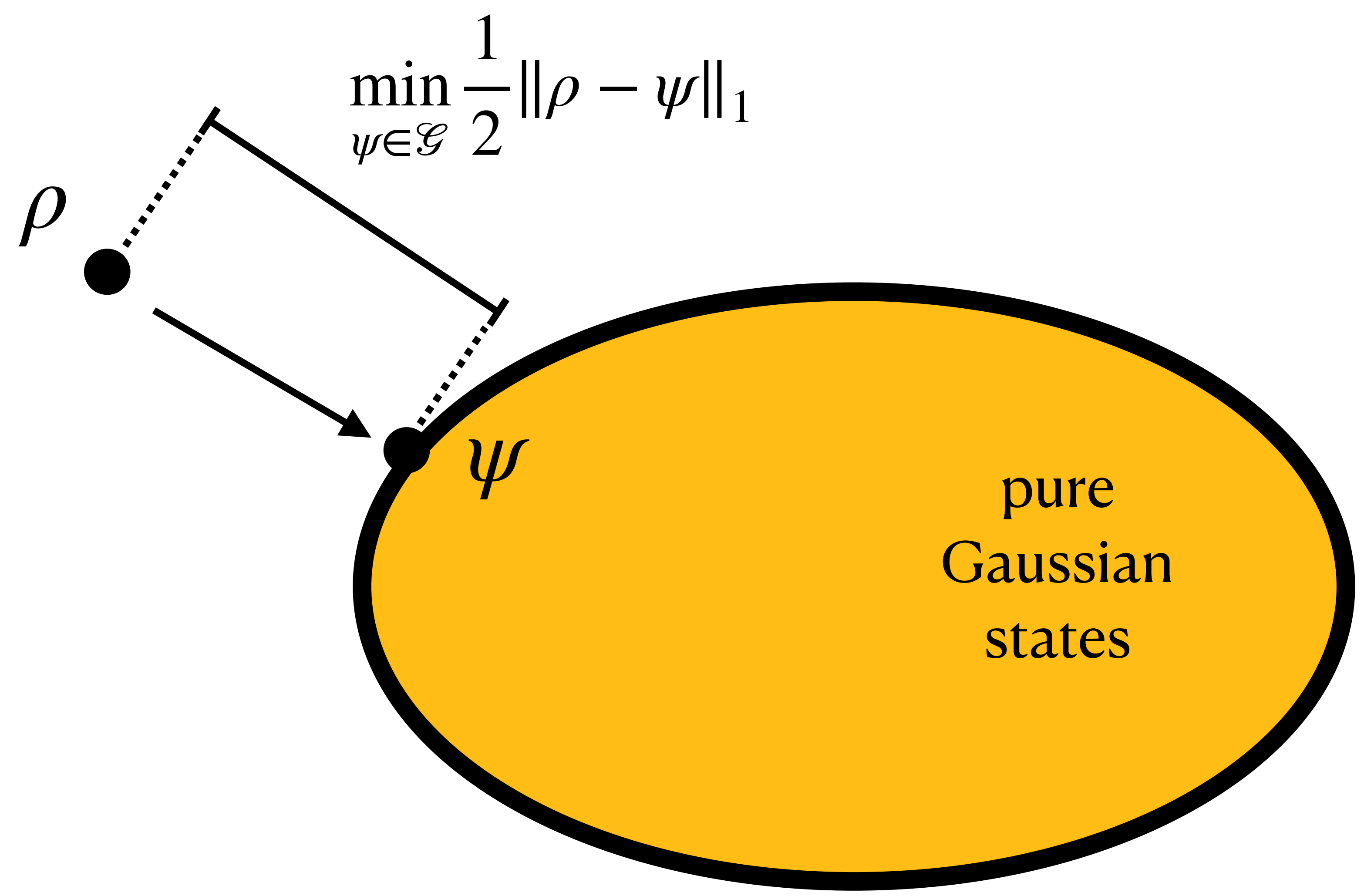
$$\frac{1}{2}\|\rho - \sigma\|_1 \ge \frac{\|\mathbf{m} - \mathbf{t}\|^2}{32\max\left(\mathrm{Tr}[\hat{E}\rho], \mathrm{Tr}\hat{E}\sigma]\right)}$$
$$\frac{1}{2}\|\rho - \sigma\|_1 \ge \frac{\|V - W\|_\infty^2}{3098\,\max(\mathrm{Tr}[\hat{E}^2\rho], \mathrm{Tr}[\hat{E}^2\sigma])}$$

Two approaches

symmetries
of Gaussian states

learning
Gaussian states

$\rho^{\otimes N}$

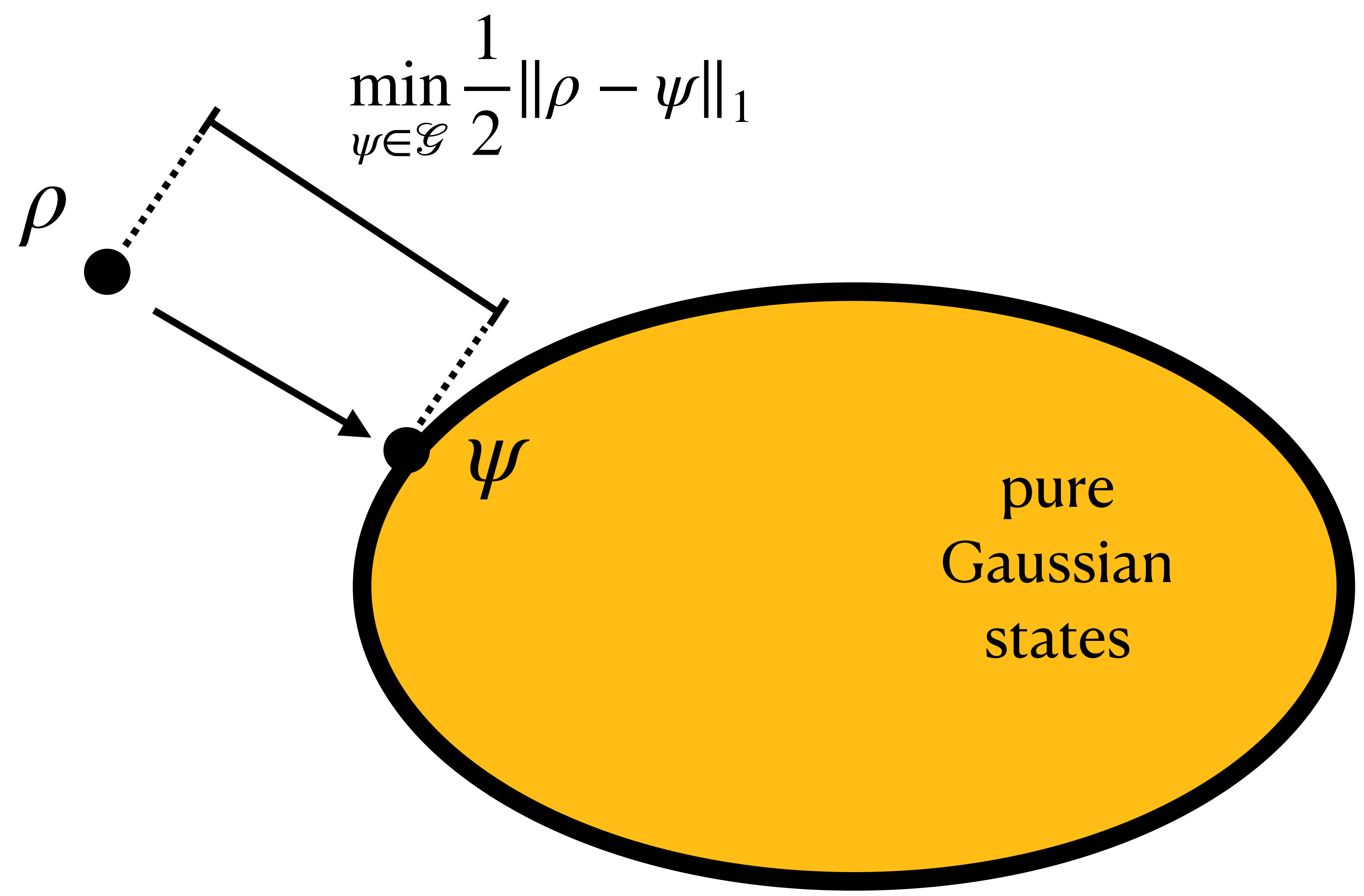


Two approaches

symmetries
of Gaussian states

$$\rho^{\otimes N} \longrightarrow m, V = SDS^\intercal$$

learning
Gaussian states



Two approaches

symmetries
of Gaussian states

learning
Gaussian states

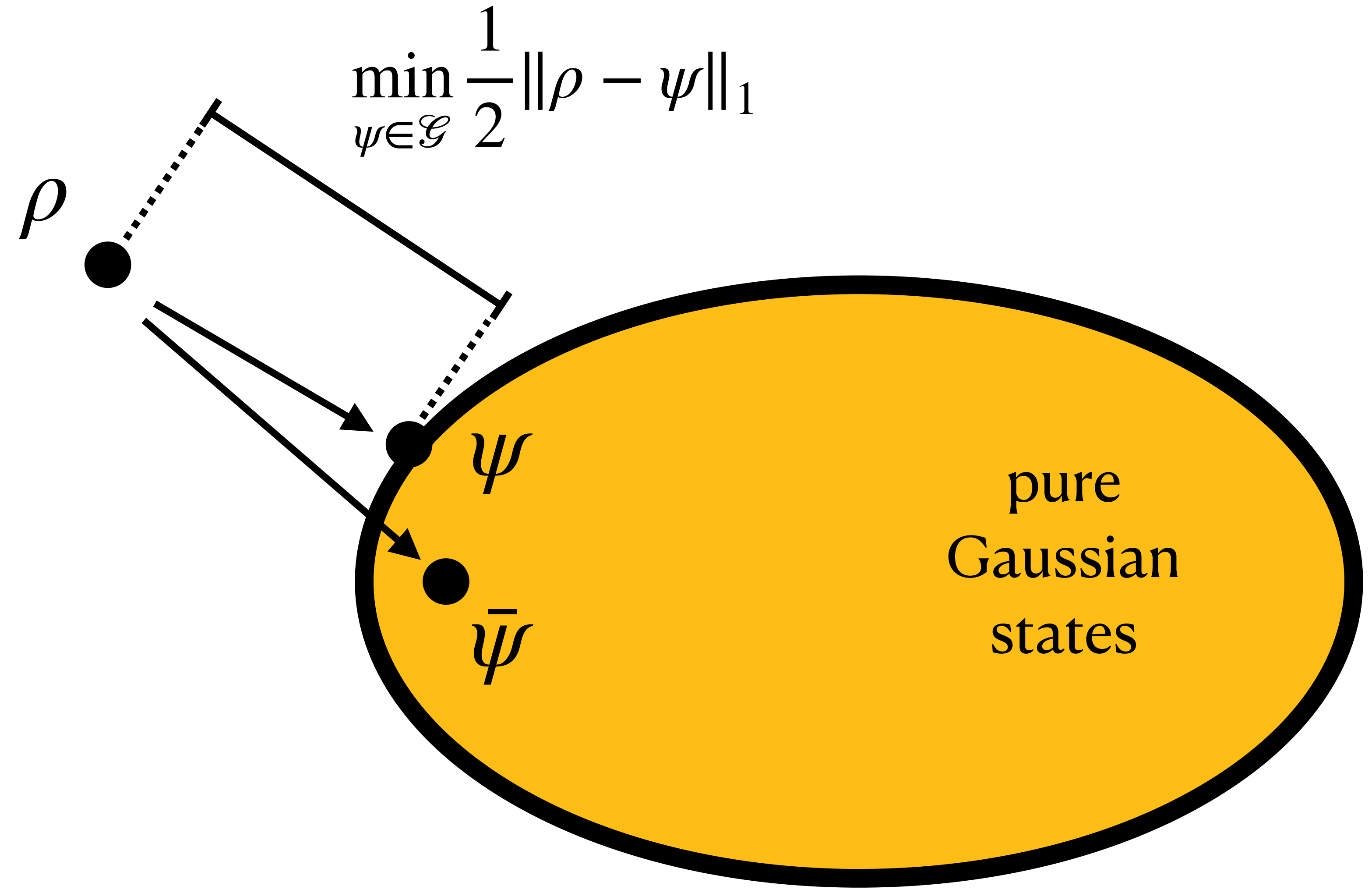
$$\rho^{\otimes N}$$

$$\downarrow$$

$$m, V = SDS^\intercal$$

$$\downarrow$$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^\intercal$



Two approaches

symmetries
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learning
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$$\rho^{\otimes N}$$
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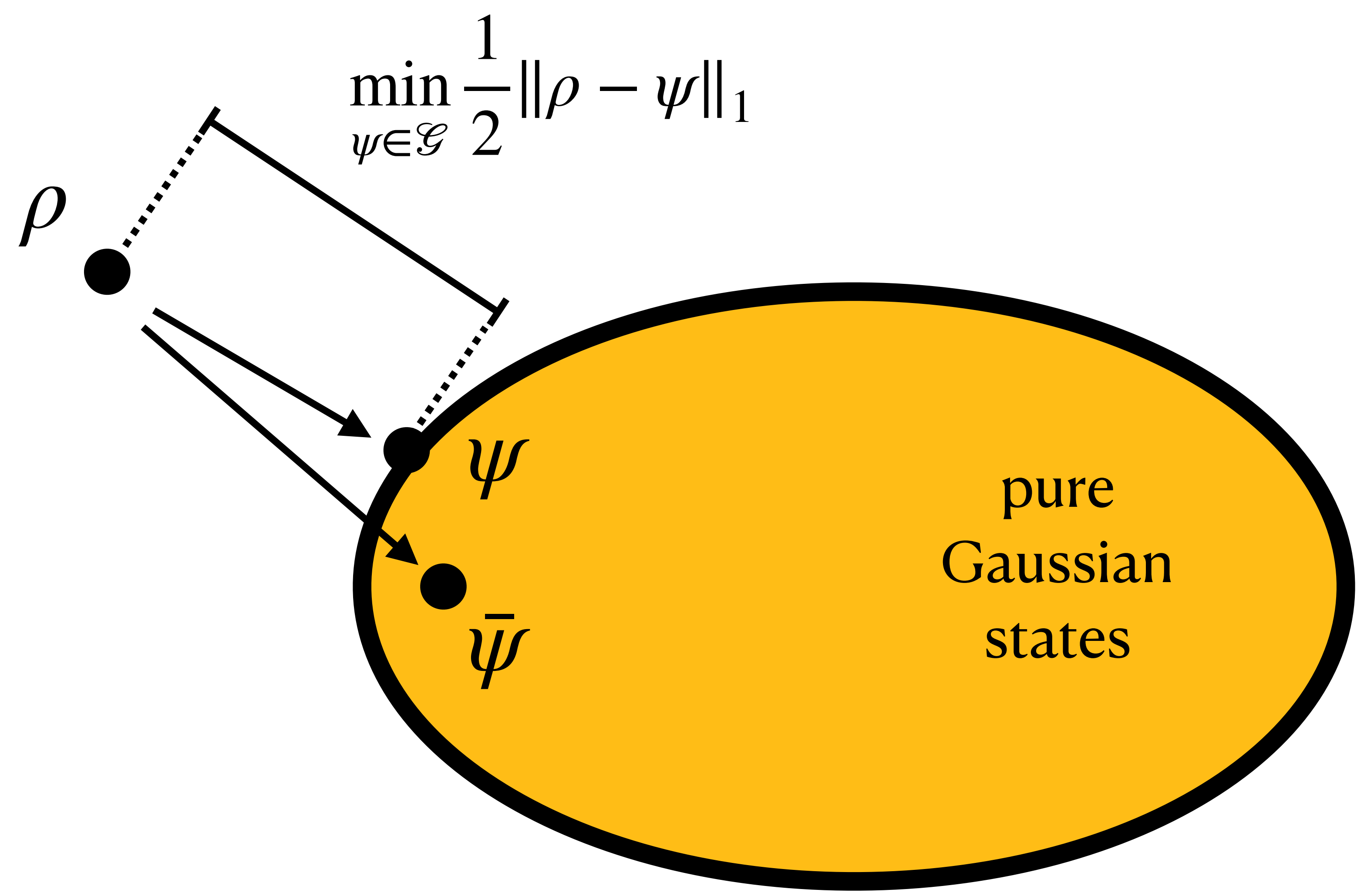

$$\min_{\psi \in \mathcal{G}_E} \frac{1}{2} \|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (\nu_i - 1)},$$

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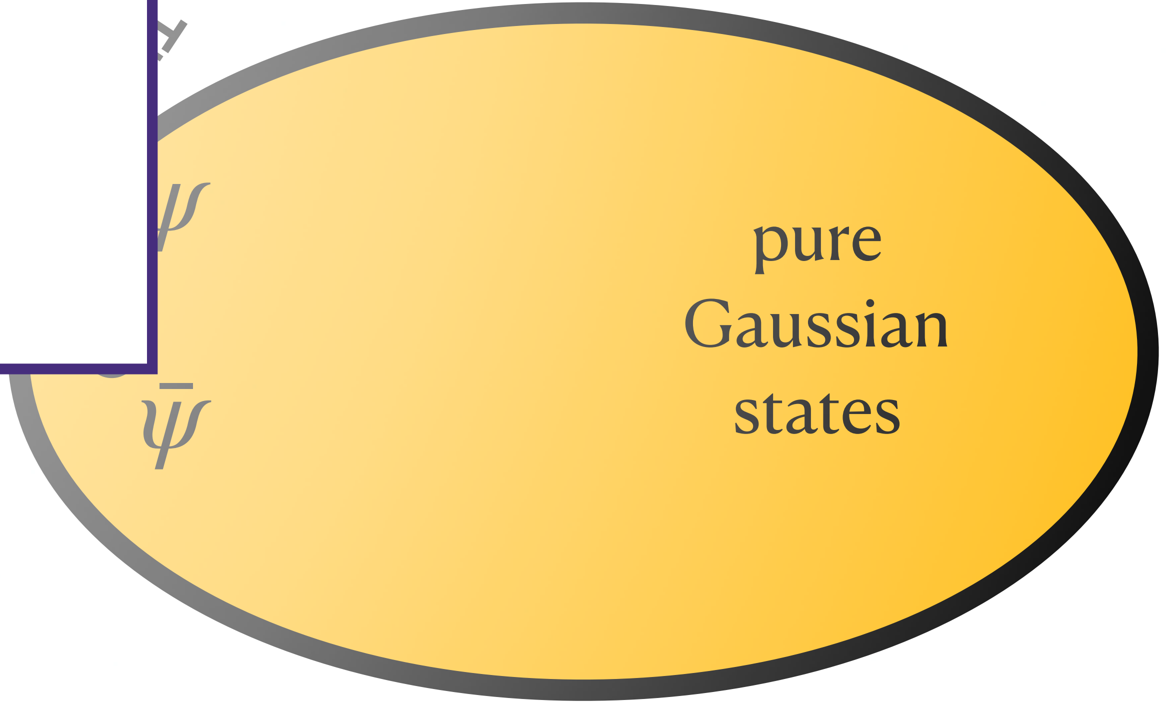
$$\bar{\psi} = D_{m(\rho)} U_S |0\rangle\langle 0| U_S^\dagger D_{m(\rho)}^\dagger$$

$$\begin{aligned}
\frac{1}{2}\|\rho - \bar{\psi}\|_1 &\overset{(i)}{\leq} \sqrt{1 - \mathrm{Tr}[\rho\bar{\psi}]} \\
&= \sqrt{1 - \mathrm{Tr}[U_S^\dagger D_{m(\rho)}^\dagger \rho D_{m(\rho)} U_S\, |0\rangle\langle 0|]} \\
&\overset{(ii)}{\leq} \sqrt{\mathrm{Tr}[U_S^\dagger D_{m(\rho)}^\dagger \rho D_{m(\rho)} U_S\, \hat{N}]} \\
&\overset{(iii)}{=} \frac{1}{2}\sqrt{\mathrm{Tr}[D - \mathbb{1}]} \\
&= \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n}(\nu_i - 1)}\,.
\end{aligned}$$

$\mathbb{1} - |0\rangle\langle 0| \leq \hat{N}$

$$\frac{1}{2}\|\rho - \psi\|_1$$



$$\rho^{\otimes N}$$

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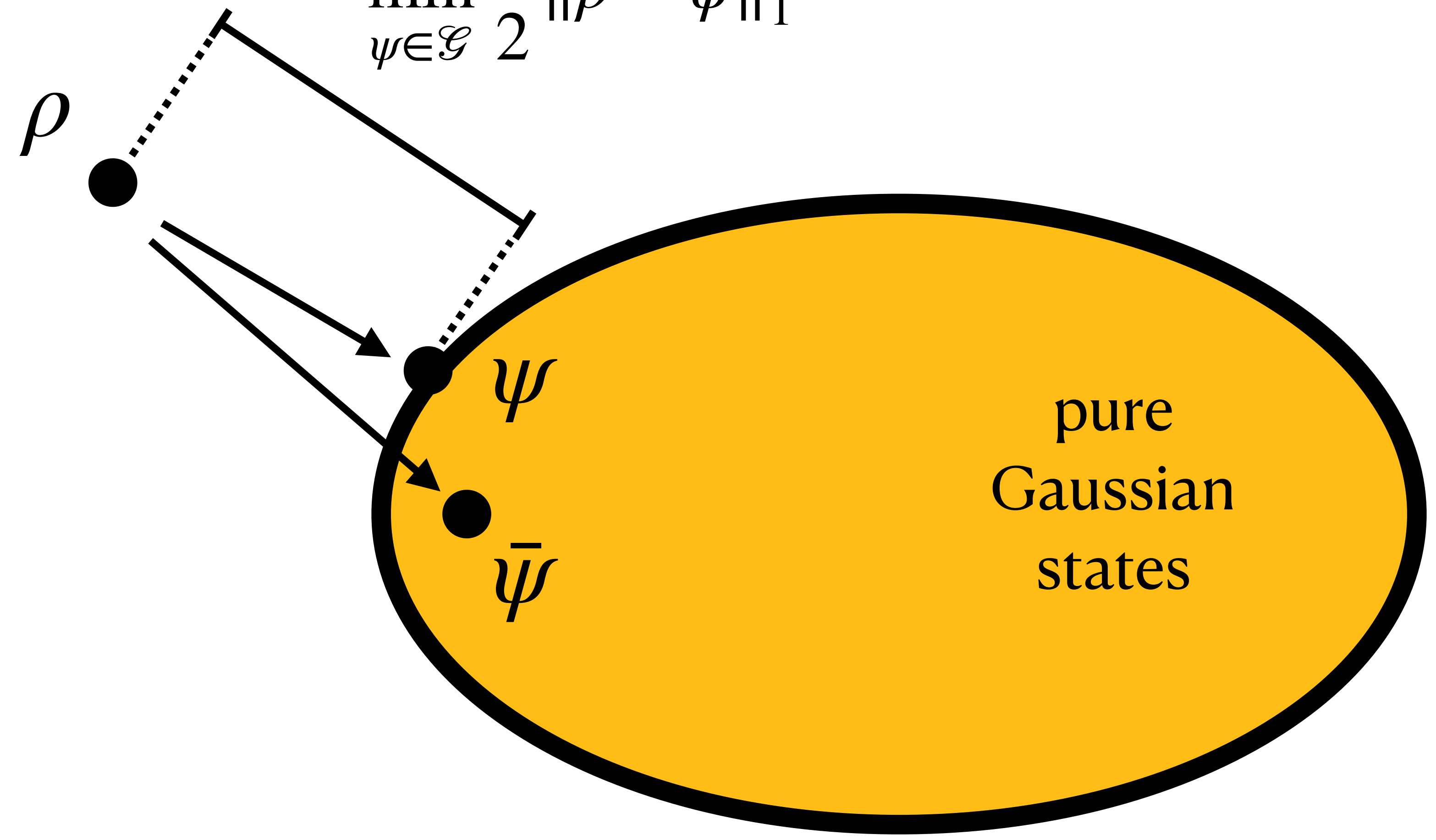
$$\min_{\psi \in \mathcal{G}} \frac{1}{2} \|\rho - \psi\|_1$$



$$\rho^{\otimes N}$$

$$\downarrow$$

$$m, V = SDS^{\intercal}$$

$$\downarrow$$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^{\intercal}$

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Mele & al. Nature Physics 21, 2002-2008 (2025)
This procedure only requires single-copy measurements.

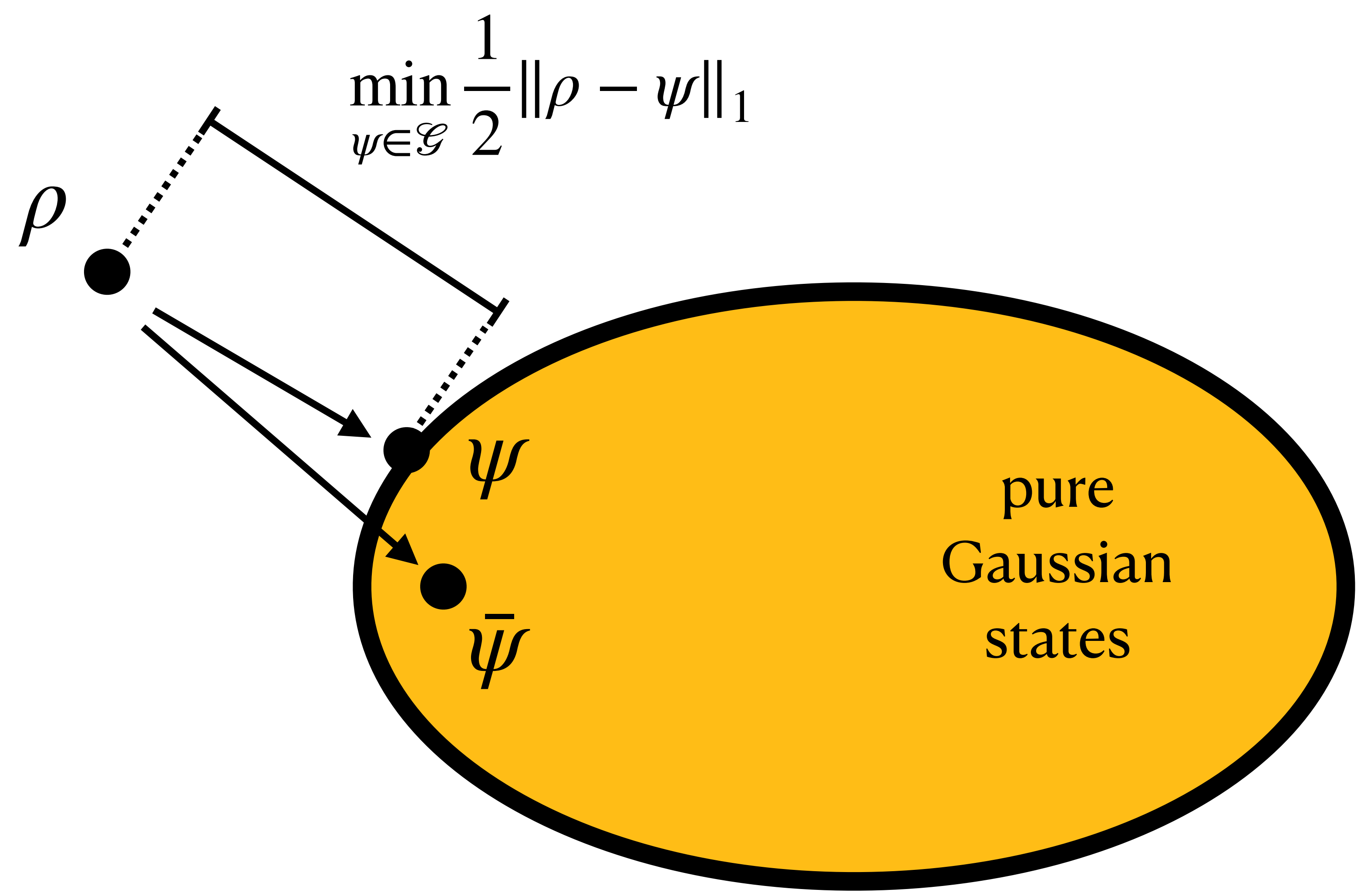
$\rho^{\otimes N} \longrightarrow$  $= \tilde{V} = \tilde{\nu}_{\max}$

$\downarrow$

$m, V = SDS^\intercal$

$\downarrow$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^\intercal$



$$\min_{\psi \in \mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n} (\nu_i - 1)},$$

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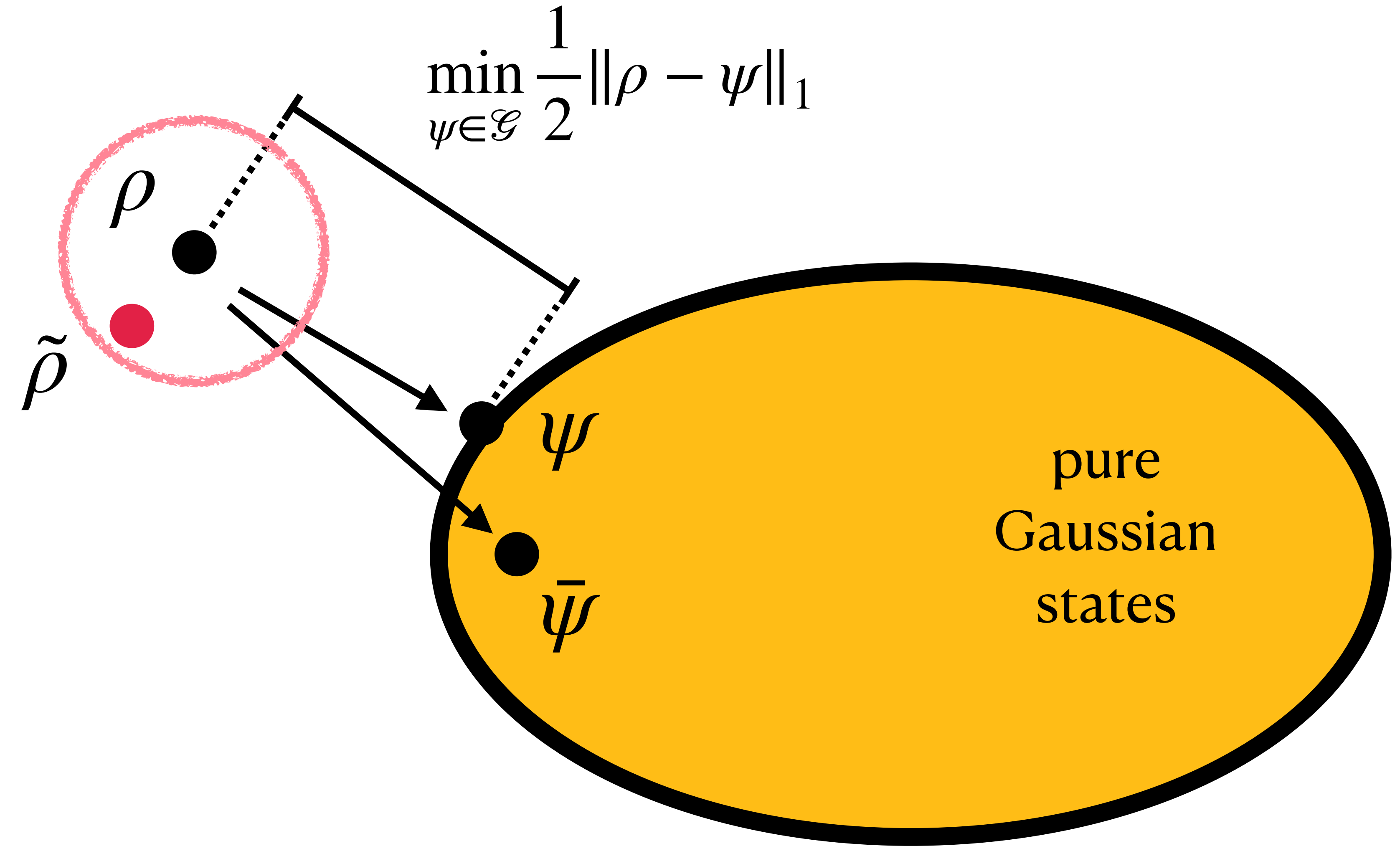
$$\boxed{\rho^{\otimes N} \longrightarrow \quad = \tilde{V} =} \tilde{\nu}_{\max}$$

$$m, V = SDS^{\intercal} \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V-\tilde{V}\|_2 \le \varepsilon_V$$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^{\intercal}$

$\tilde{\rho}$ state with mean m and covariance $\tilde{V}$



$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \le \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n}(\nu_i-1)},$$

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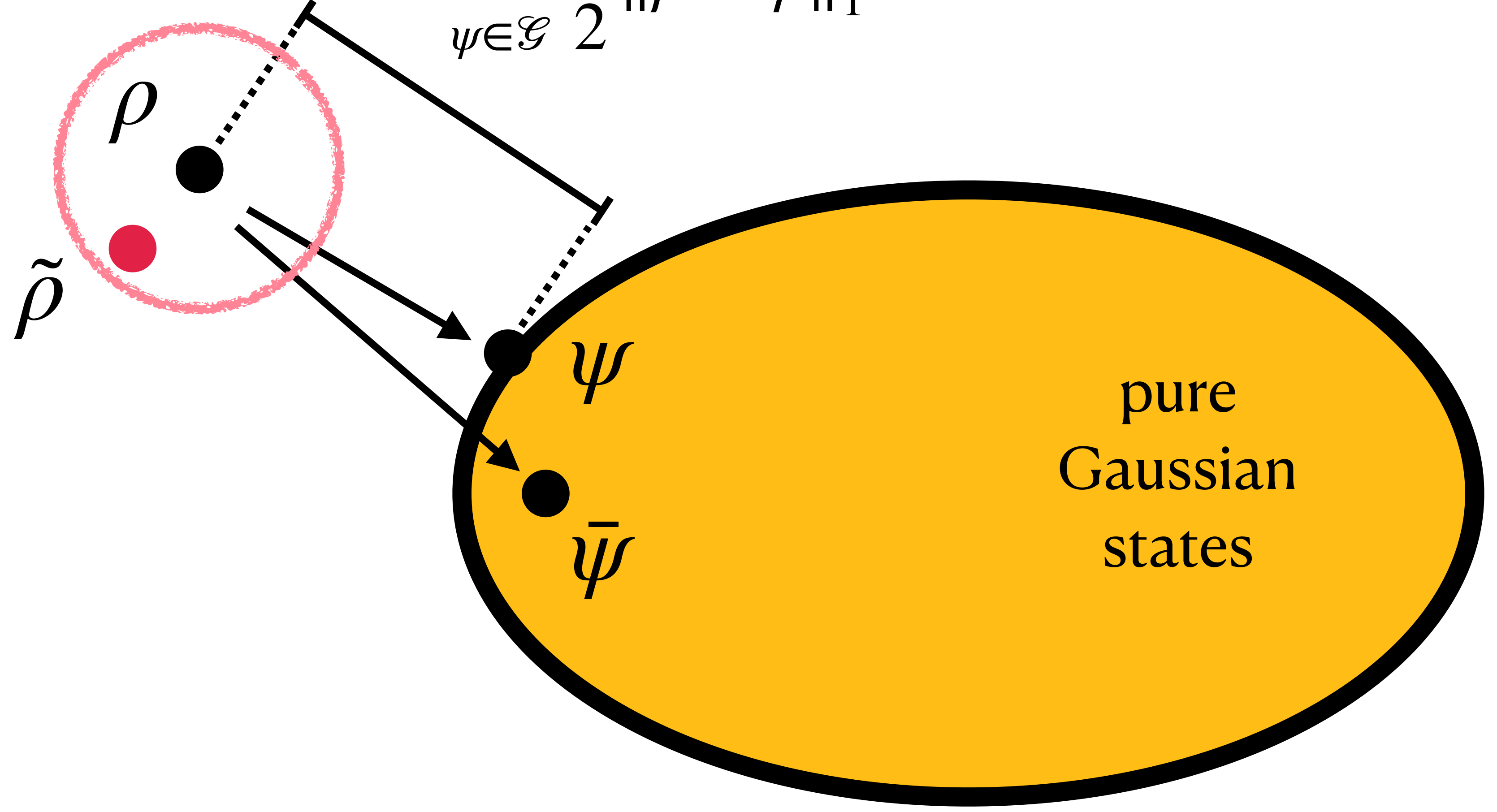
$\bar{\psi}$ pure Gaussian state with mean m
and covariance $\bar{V} = SS^\intercal$

$\tilde{\rho}$ state with mean m
and covariance $\tilde{V}$

$$\min_{\psi \in \mathcal{G}} \frac{1}{2}\|\rho - \psi\|_1$$



$$\min_{\psi \in \mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE + \varepsilon_V)\varepsilon_V}$$

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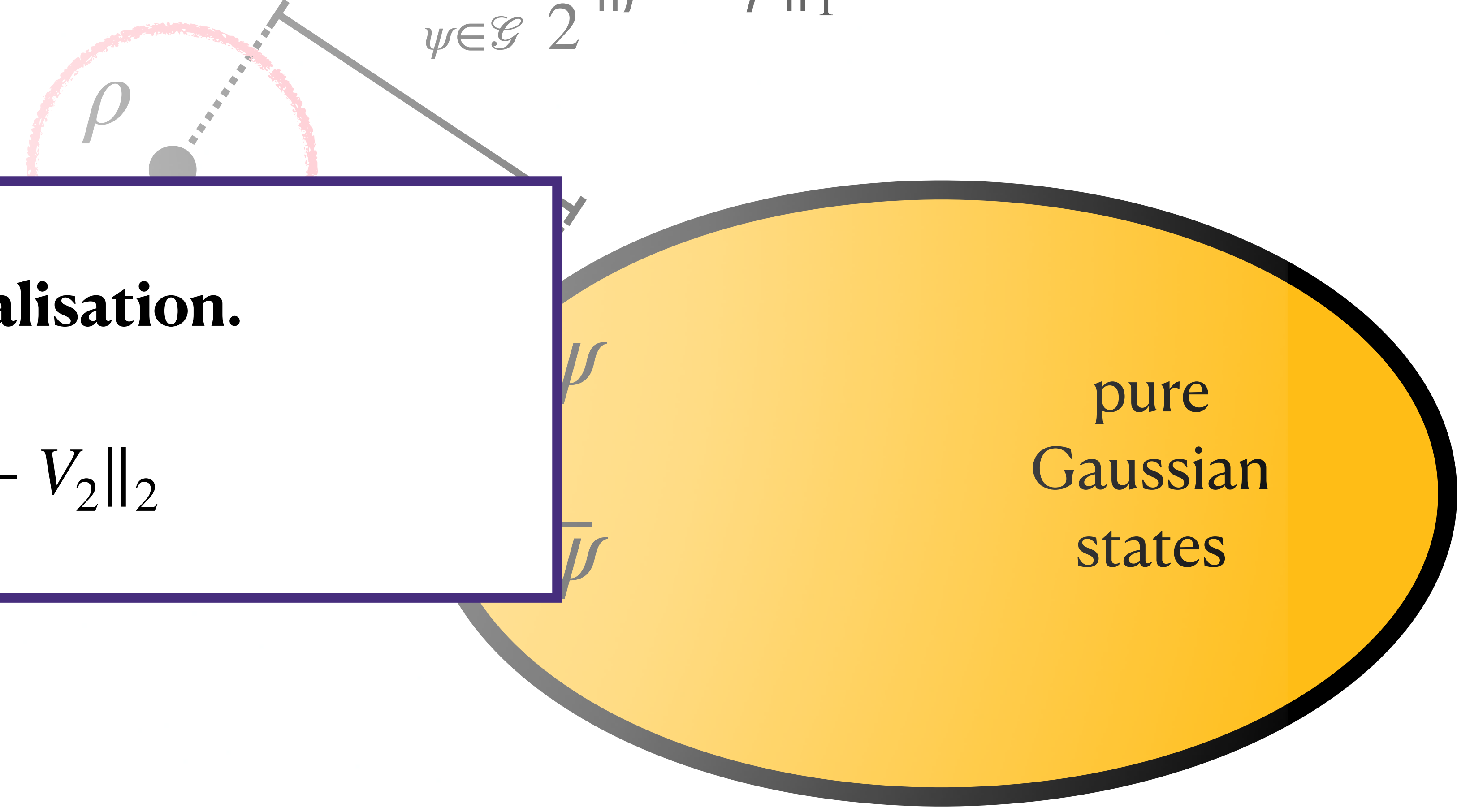
$\rho^{\otimes N} \longrightarrow \quad = \tilde{V}$

$m, V = SDS^\intercal \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V - \tilde{V}\|_2 \leq \varepsilon_V$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^\intercal$

$\tilde{\rho}$ state with mean m and covariance $\tilde{V}$

$\min_{\psi \in \mathcal{G}} \frac{1}{2}\|\rho - \psi\|_1$



Perturbation on symplectic diagonalisation.

$$\|D_1 - D_2\|_2 \leq \sqrt{K(V_1)K(V_2)}\|V_1 - V_2\|_2$$

$$\min_{\psi \in \mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE + \varepsilon_V)\varepsilon_V}$$

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$$m, V = SDS^\intercal \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V - \tilde{V}\|_2 \leq \varepsilon_V$$

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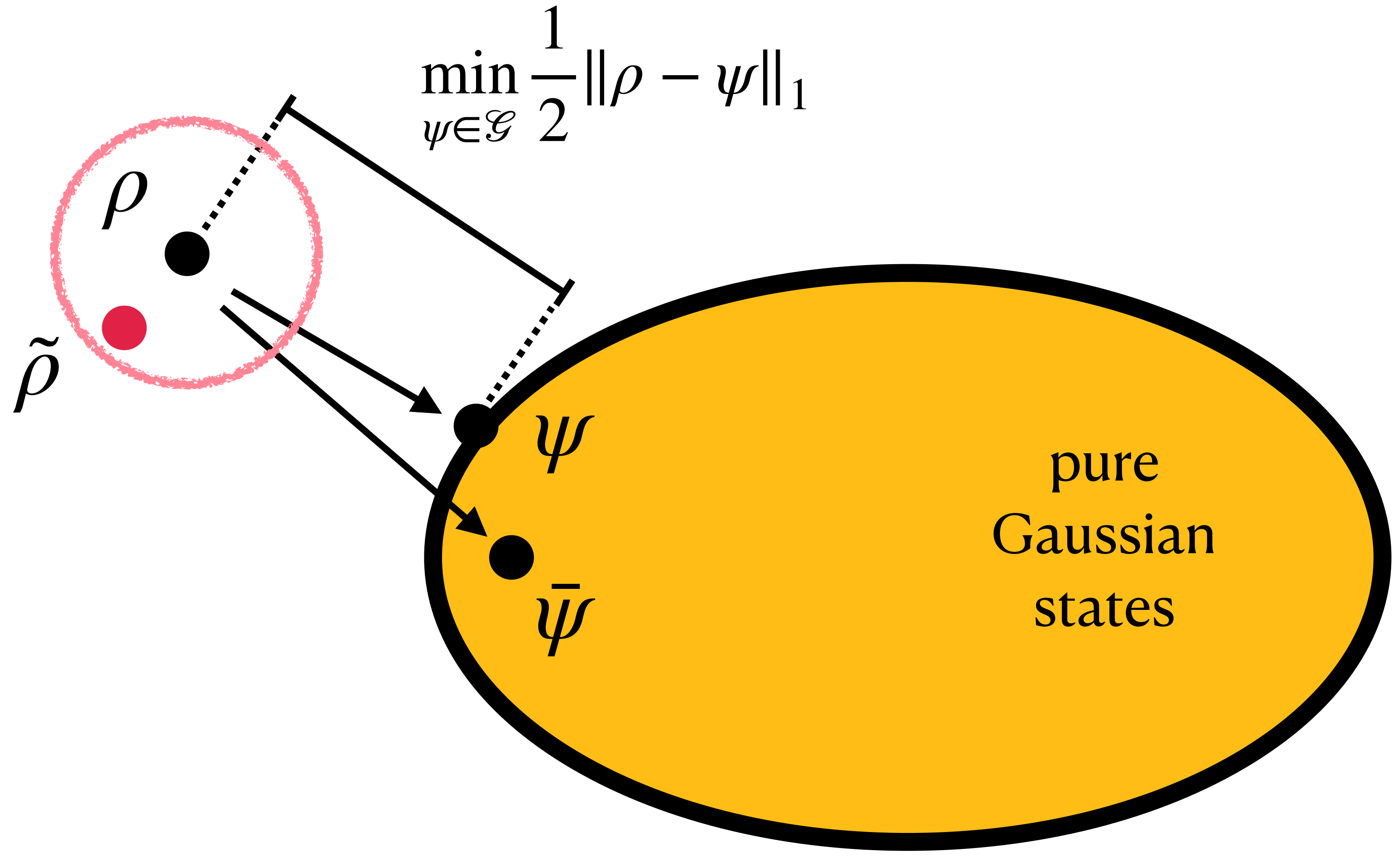

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE+\varepsilon_V)\varepsilon_V}$$

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$$\boxed{\rho^{\otimes N} \longrightarrow \quad = \tilde{V}} = \tilde{\nu}_{\max}$$

$$\downarrow$$

$$m, V = SDS^{\intercal} \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V - \tilde{V}\|_2 \leq \varepsilon_V$$

$$\downarrow$$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^{\intercal}$

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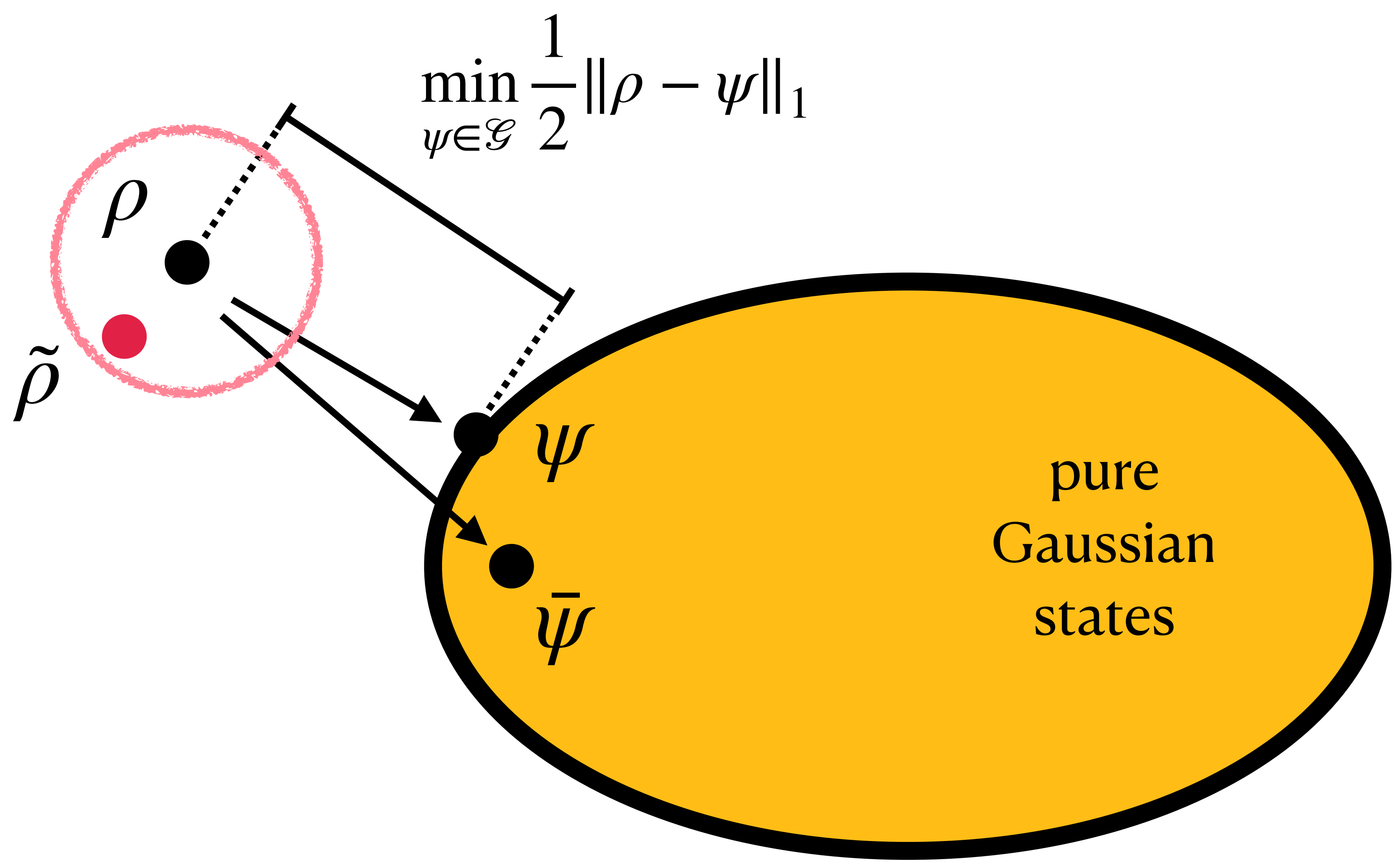

$$\min_{\psi\in\mathscr{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3 4E(4nE + \varepsilon_V)\varepsilon_V}}$$

$$\min_{\psi\in\mathscr{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \geq \frac{(\nu_{\max} - 1)^2}{c\,(nE)^6}$$

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$\rho^{\otimes N} \longrightarrow$  $= \tilde{V} =$

$m, V = SDS^{\intercal}$ $\quad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right)$

$\bar{\psi}$ pure Gaussian state with mean m and covariance $\bar{V} = SS^{\intercal}$

$\tilde{\rho}$ state with mean m and covariance $\tilde{V}$

Lower bound between arbitrary states.

$$\frac{1}{2}\|\rho-\sigma\|_1 \geq \frac{\|V-W\|_\infty^2}{3098\ \max(\mathrm{Tr}[\hat{E}^2\rho], \mathrm{Tr}[\hat{E}^2\sigma])}$$

$$\min_{\psi\in\mathcal{G}_E}\frac{1}{2}\|\rho-\psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE+\varepsilon_V)\varepsilon_V}$$

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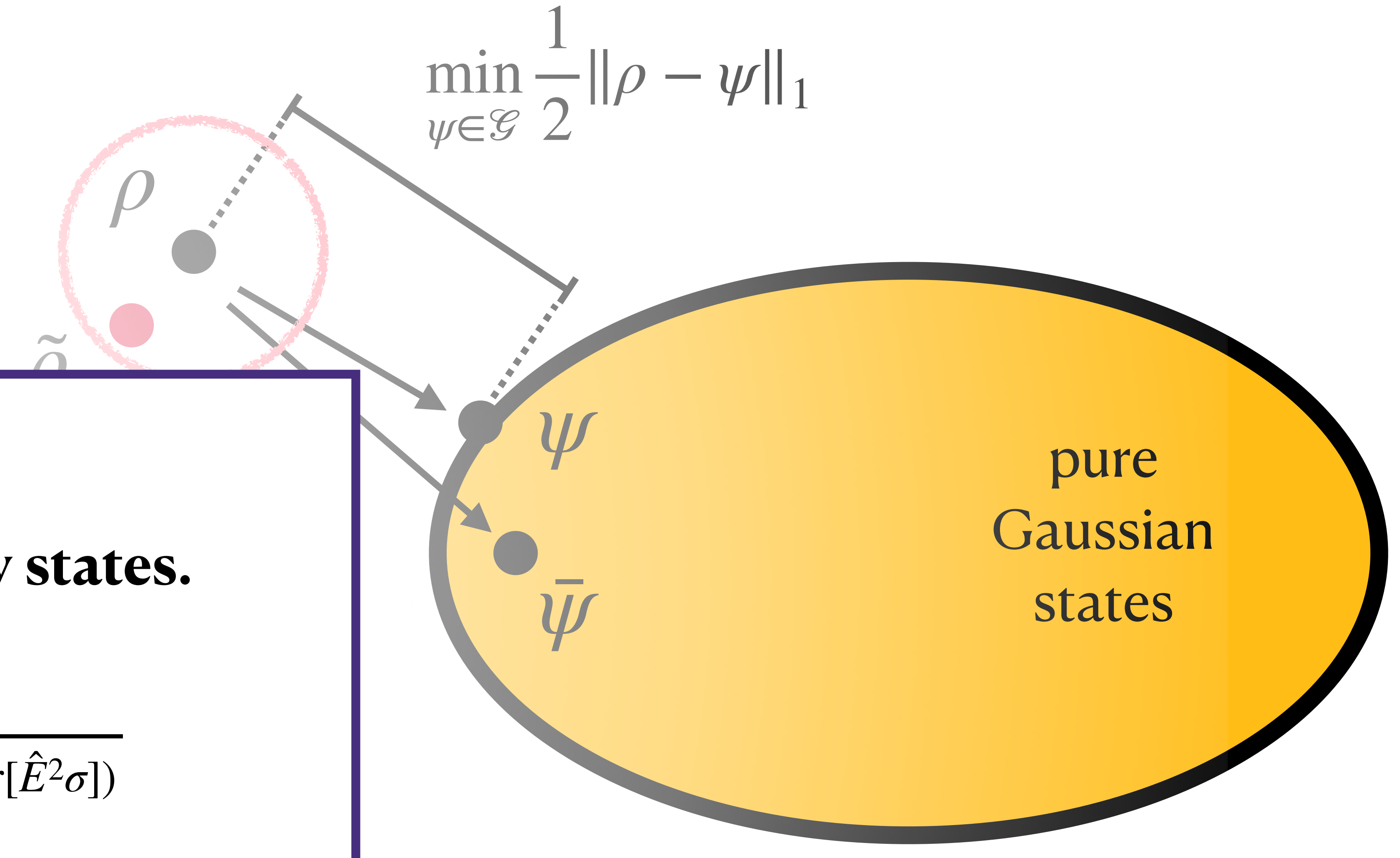

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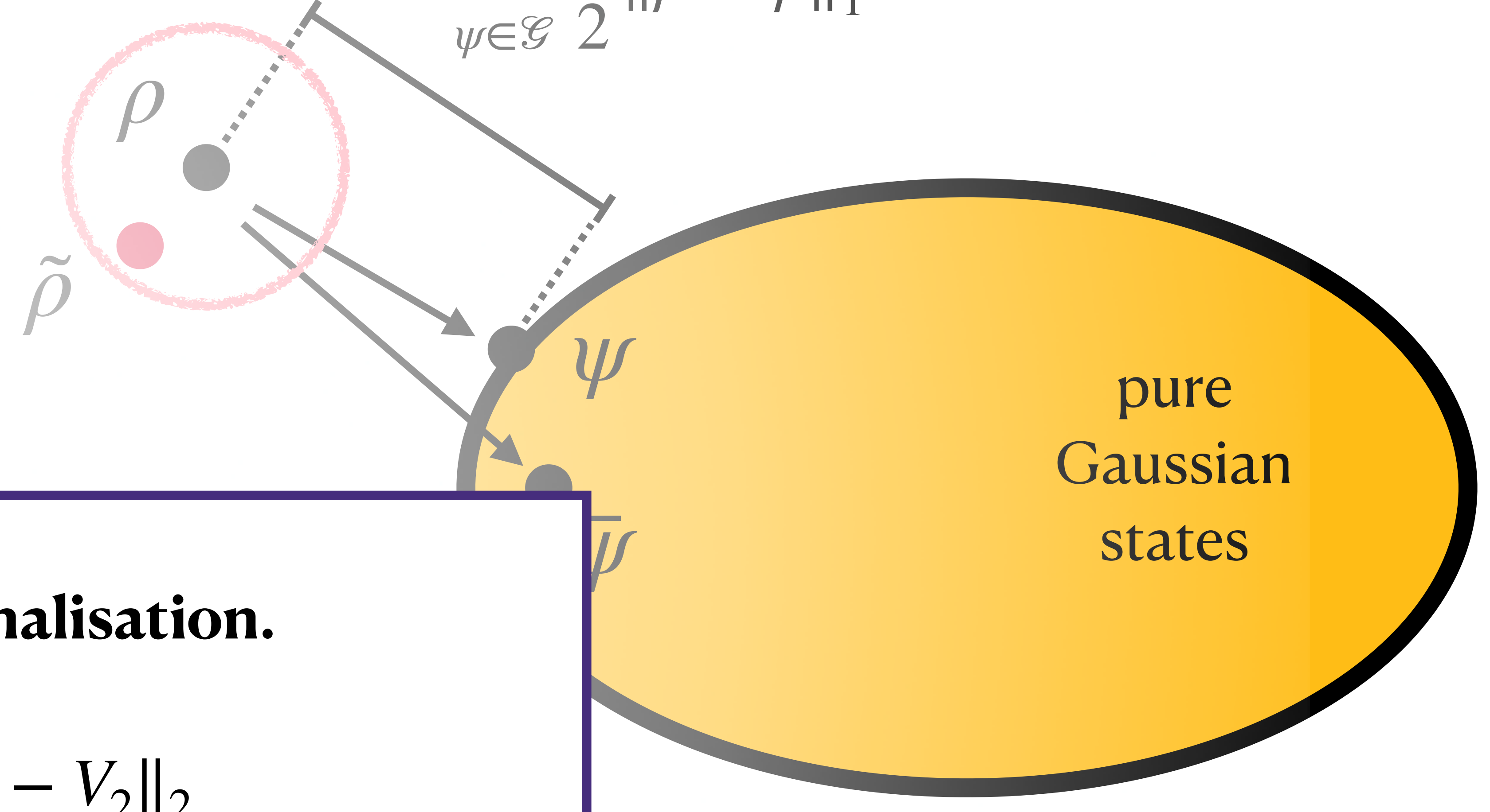
$$\min_{\psi \in \mathcal{G}} \frac{1}{2}\|\rho - \psi\|_1$$



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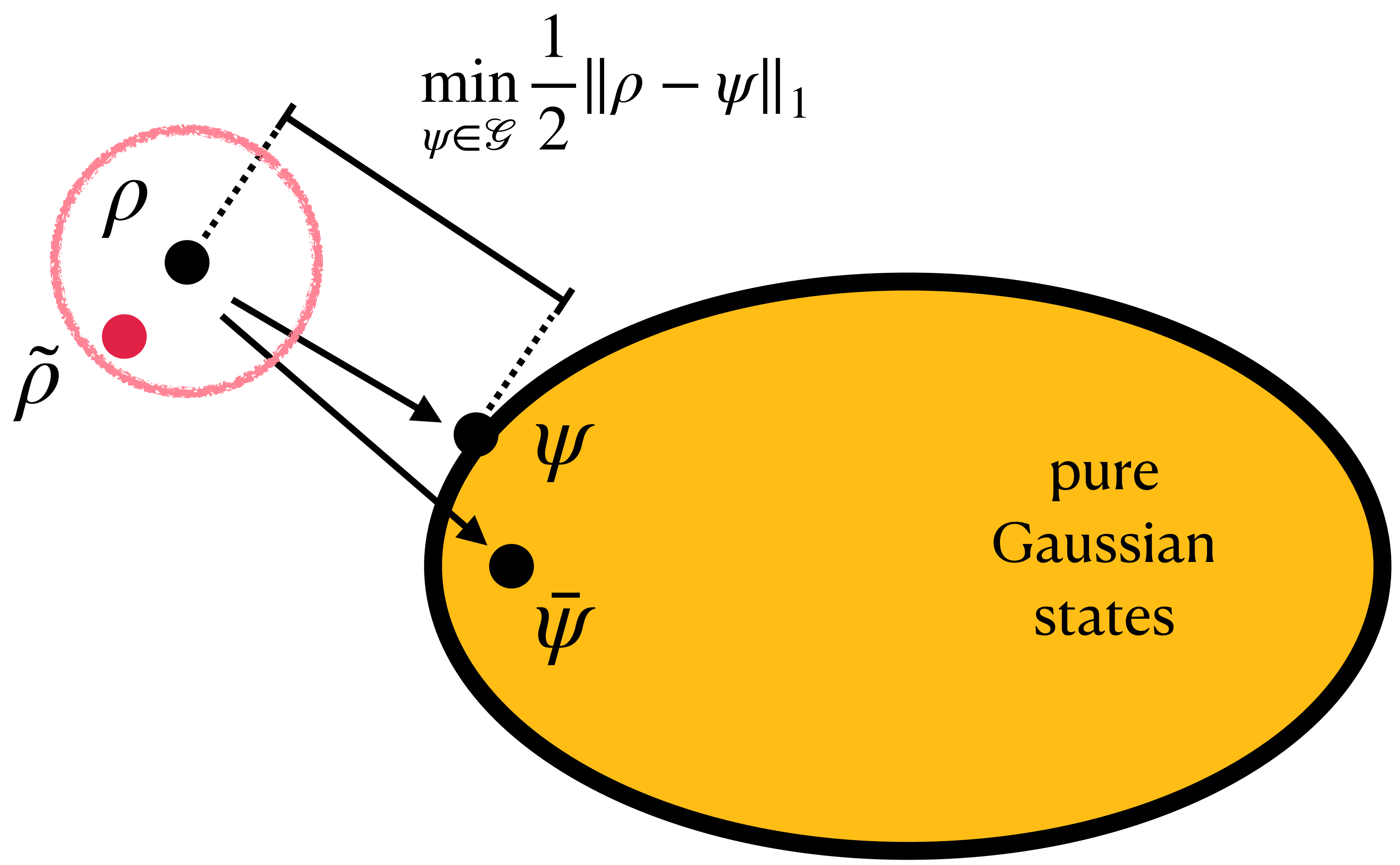

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3 4E(4nE+\varepsilon_V)\varepsilon_V}}$$

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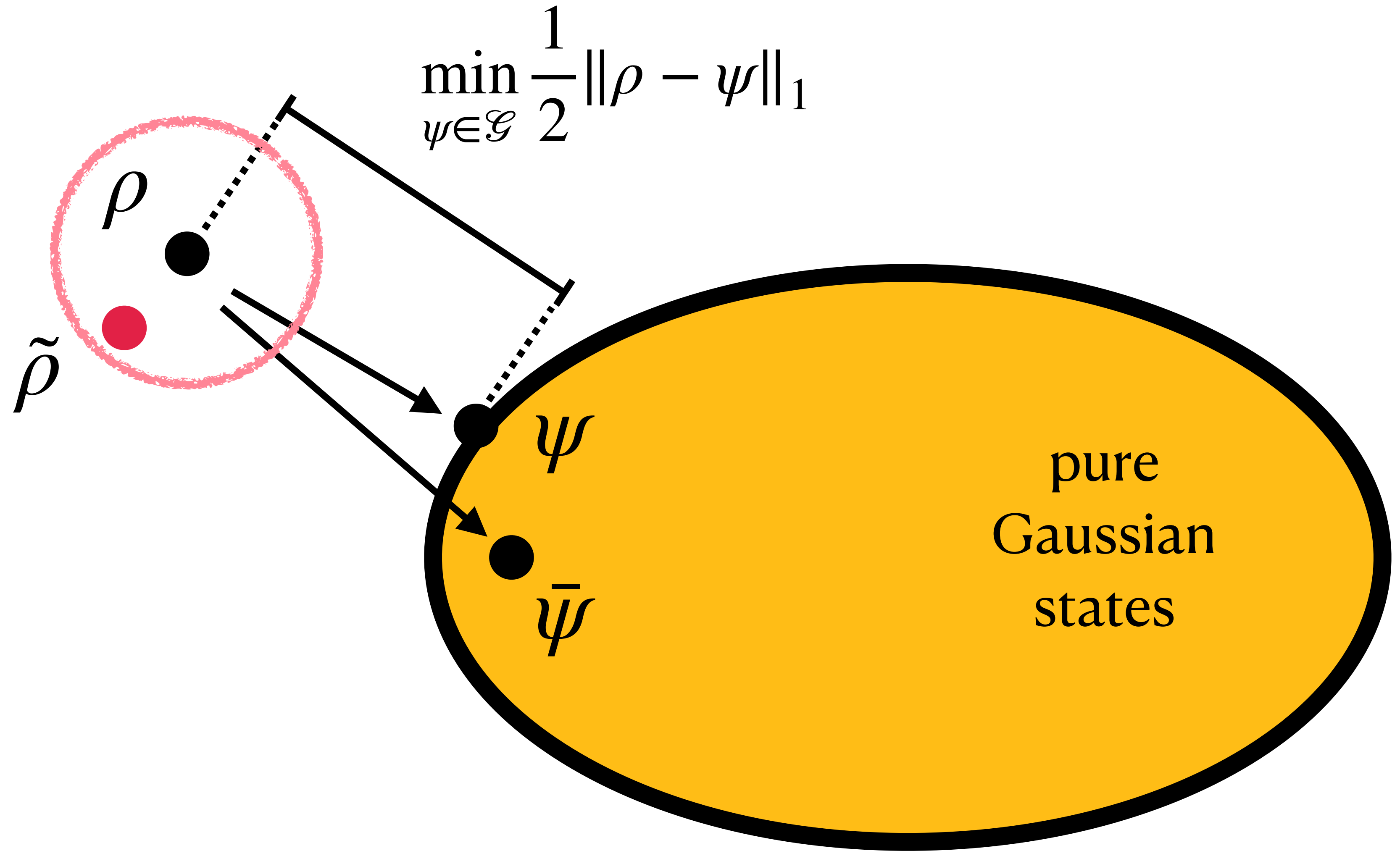

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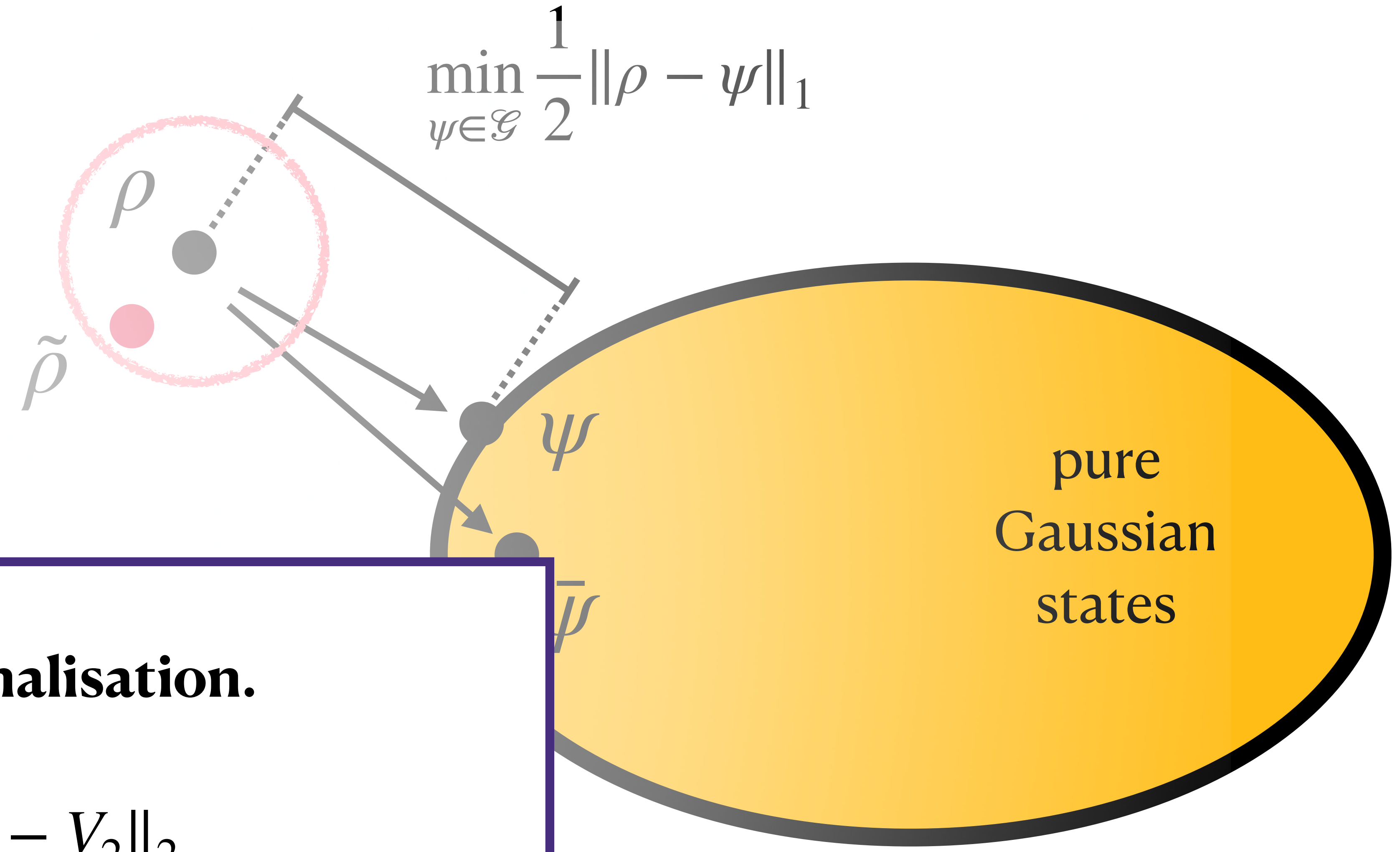

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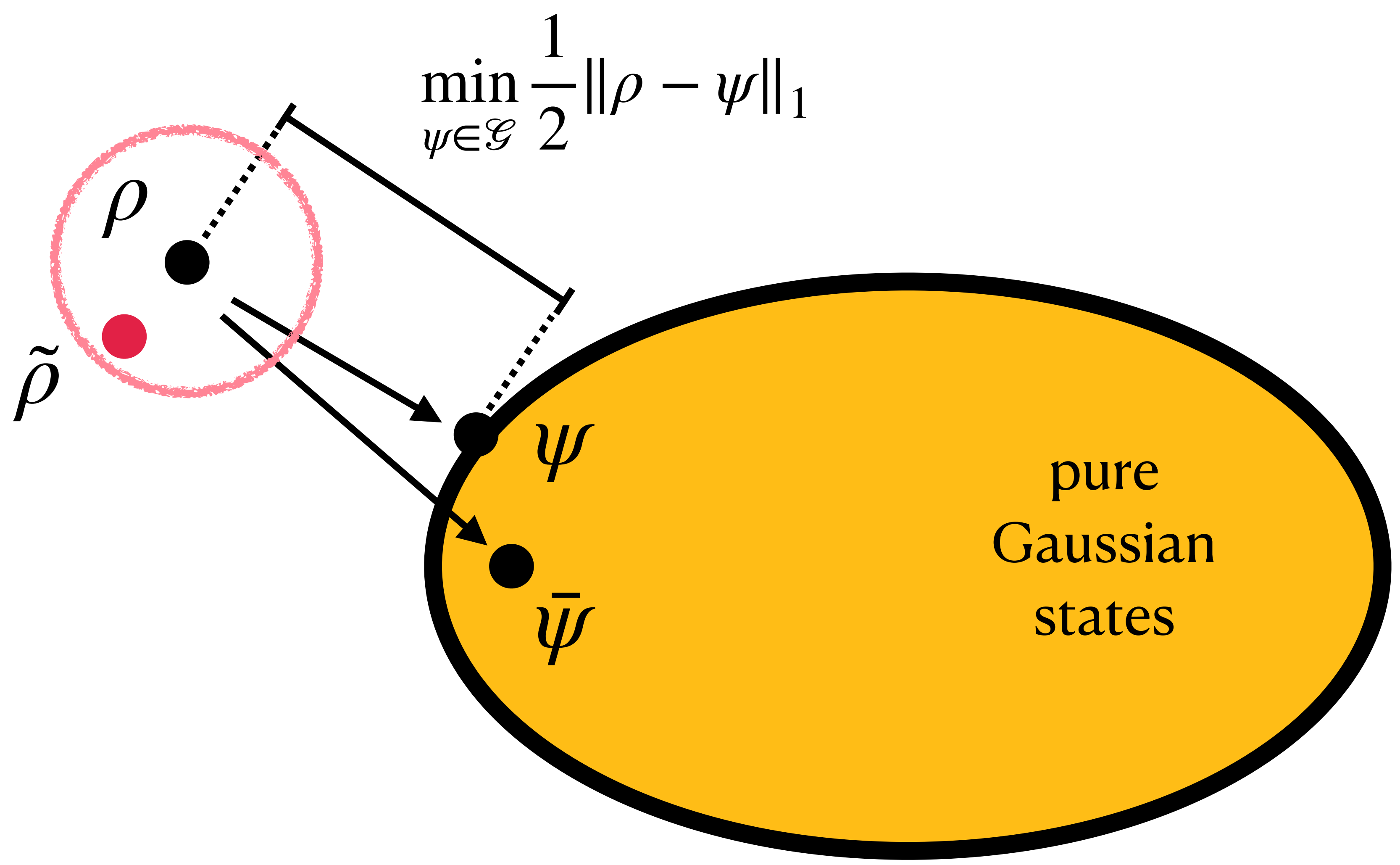

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$$\rho^{\otimes N} \longrightarrow \quad = \tilde{V} = \tilde{\nu}_{\max} \text{ vs } \nu_{\text{thr}}$$

$$m, V = SDS^{\intercal} \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V - \tilde{V}\|_2 \leq \varepsilon_V$$

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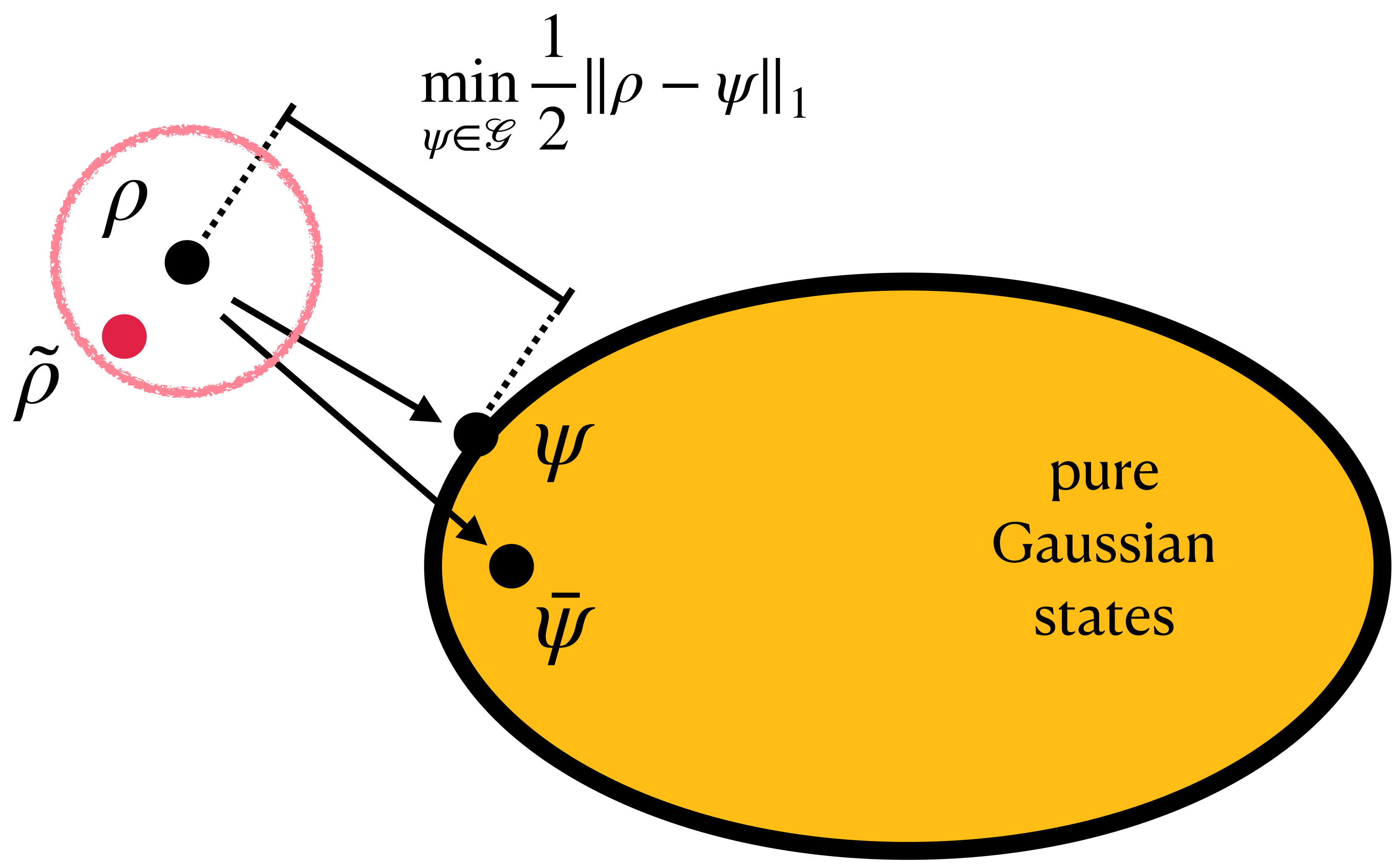

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$\rho^{\otimes N} \longrightarrow$  $= \tilde{V} = \tilde{\nu}_{\max}$ vs ν_{thr}

$\leq$

$m, V = SDS^\intercal$ $\quad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right)$ $\quad \|V - \tilde{V}\|_2 \leq \varepsilon_V$

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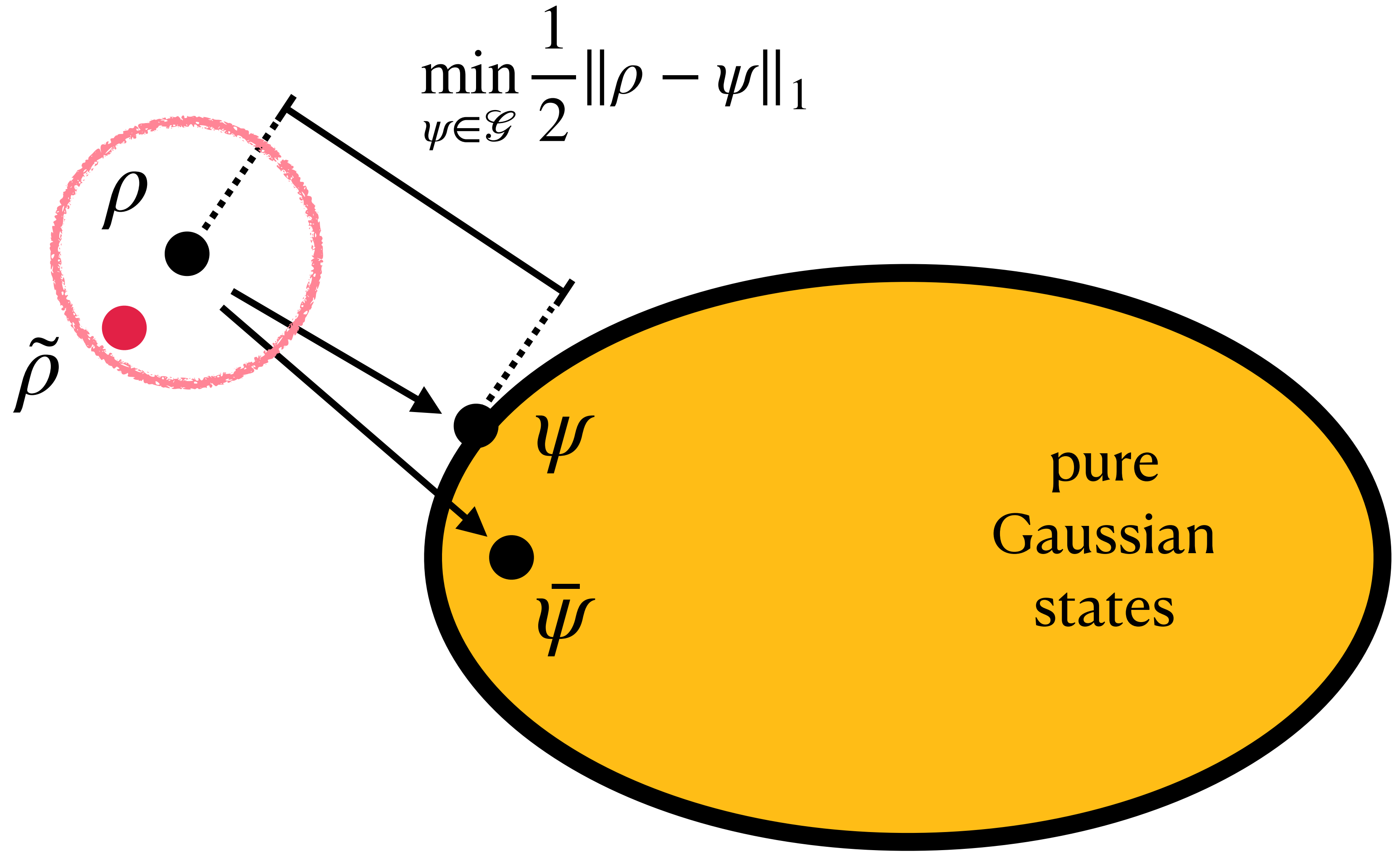

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE+\varepsilon_V)\varepsilon_V} \leq \varepsilon_B$$

P **G**

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho - \psi\|_1 \geq \frac{1}{2c\,(nE)^6}\left[(\tilde{\nu}_{\max} - 1)^2 - 8nE(4nE+\varepsilon_V)\varepsilon_V\right]$$

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$\rho^{\otimes N} \longrightarrow$  $= \tilde{V} = \tilde{\nu}_{\max}$ vs ν_{thr} ($\leq$ / $>$)

$m, V = SDS^\intercal \qquad N = O\left(\log\left(\frac{n^2}{\delta}\right)\frac{n^3E^2}{\varepsilon_V^2}\right) \qquad \|V-\tilde{V}\|_2 \leq \varepsilon_V$

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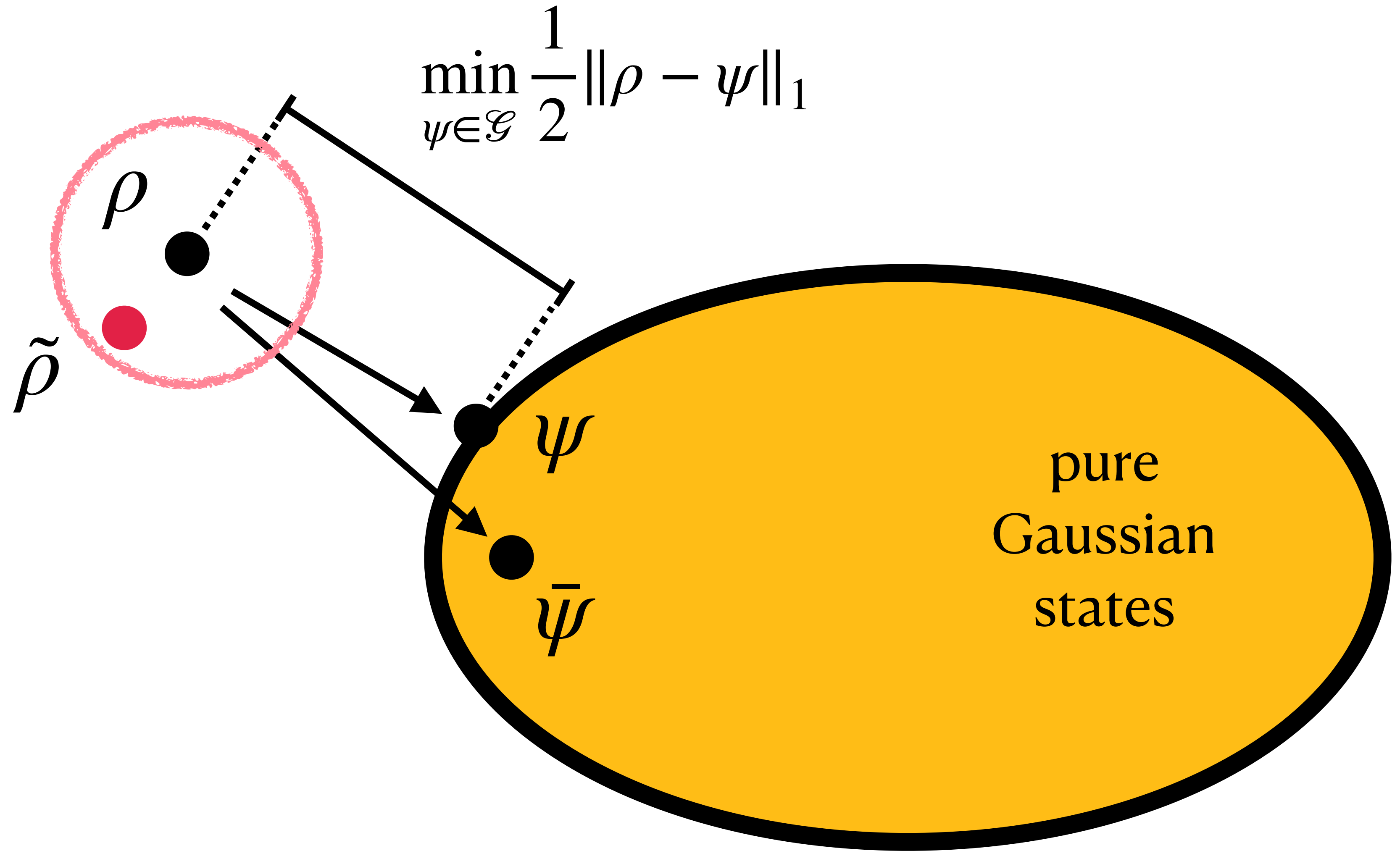

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE+\varepsilon_V)\varepsilon_V} \leq \varepsilon_B \quad \text{P} \quad \text{G}$$

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \geq \frac{1}{2c\,(nE)^6}\left[(\tilde{\nu}_{\max}-1)^2 - 8nE(4nE+\varepsilon_V)\varepsilon_V\right] > \varepsilon_A \quad \text{G} \quad \cancel{\text{P}}$$

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Theorem. Let $0 \leq \varepsilon_A < \varepsilon_B$ and $\delta \in (0,1)$. Let ρ be a possibly mixed state satisfying the energy bound $\sqrt{\mathrm{Tr}[\hat{E}^2\rho]} \leq nE$. If $\eta := \varepsilon_B^4 - \frac{c}{2}n^8E^6\varepsilon_A > 0$, then **single-copy measurements** on

$$N = O\left(\log\left(\frac{n}{\delta}\right)\frac{n^7E^6}{\eta^2}\right) = O\big(\mathrm{poly}(n, E)\big)$$

copies of ρ suffice to decide whether ρ is ε_A-close or ε_B-far from $\mathcal{G}_E$ with success probability at least $1 - \delta$.

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learning
Gaussian states

Theorem. Let $0 \leq \varepsilon_A < \varepsilon_B$ and $\delta \in (0,1)$. Let ρ be a possibly mixed state satisfying the energy bound $\sqrt{\mathrm{Tr}[\hat{E}^2\rho]} \leq nE$. If $\eta := \boxed{\varepsilon_B^4 - \frac{c}{2}n^8E^6\varepsilon_A > 0}$, then **single-copy measurements** on

$$N = O\left(\log\left(\frac{n}{\delta}\right)\frac{n^7E^6}{\eta^2}\right) = O(\mathrm{poly}(n,E))$$

copies of ρ suffice to decide whether ρ is ε_A-close or ε_B-far from $\mathcal{G}_E$ with success probability at least $1-\delta$.

Two approaches

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Gaussian states

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Gaussian states

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$\tilde{V} \Longrightarrow \tilde{\nu}_{\max}$ vs ν_{thr}

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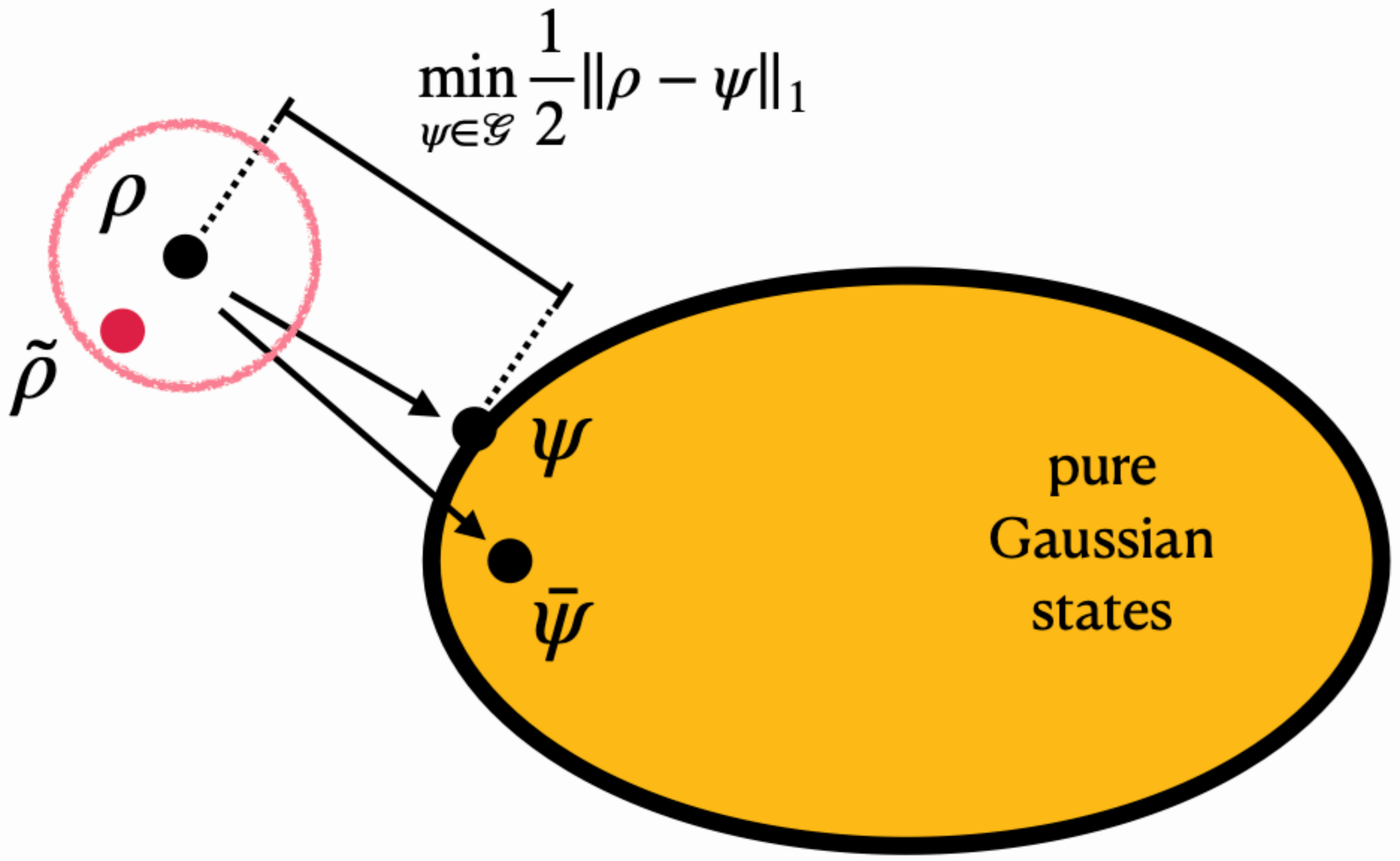
$\left(\frac{n^3E^2}{\varepsilon_V^2}\right)$ $\|V-\tilde{V}\|_2 \leq \varepsilon_V$



$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \leq \frac{1}{\sqrt{2}}\sqrt{n(\max_i \tilde{\nu}_i - 1) + \sqrt{n^3}4E(4nE+\varepsilon_V)\varepsilon_V} \leq \varepsilon_B$$

P **G**

$$\min_{\psi\in\mathcal{G}_E} \frac{1}{2}\|\rho-\psi\|_1 \geq \frac{1}{2c\,(nE)^6}\left[(\tilde{\nu}_{\max}-1)^2 - 8nE(4nE+\varepsilon_V)\varepsilon_V\right] > \varepsilon_A$$

G ~~**P**~~

Two approaches

symmetries
of Gaussian states

learning
Gaussian states

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5 minutes break

Mixed Gaussian states

Definition of the problem

Let $0 \leq \varepsilon_A < \varepsilon_B$, let $0 < \delta \leq 1$ and let

- $\mathcal{G}_E^{\text{mixed}}$ be the set of **mixed Gaussian states** with mean energy per mode at most E.

We say that an algorithm solves the property testing problem using N samples if, for any generic state ρ such that

A. either ρ is ε_A-close to $\mathcal{G}_E^{\text{mixed}}$, i.e. $\boxed{\min_{\sigma \in \mathcal{G}_E^{\text{mixed}}} \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon_A}$ **P**

B. or ρ is ε_B-far from $\mathcal{G}_E^{\text{mixed}}$, i.e. $\boxed{\min_{\sigma \in \mathcal{G}_E^{\text{mixed}}} \frac{1}{2}\|\rho - \sigma\|_1 > \varepsilon_B}$, $\cancel{\text{P}}$

the algorithm can identify the underlying hypothesis A or B with failure probability at most δ.

Hardness result

Theorem. Let ρ be a state satisfying the energy bound $\sqrt{\mathrm{Tr}[\hat{E}^2\rho]} \leq nE$. If $\varepsilon_A < \varepsilon_B < O\left(\frac{1}{\mathrm{poly}(nE)}\right)$, deciding whether ρ is ε_A-close or ε_B-far from $\mathscr{G}_E^{\mathrm{mixed}}$ with success probability larger than 2/3 requires at least

$$N = \Omega\left(\frac{E^n}{n^4 E^4 \varepsilon_B^2}\right)$$

copies of ρ.

Strategy.

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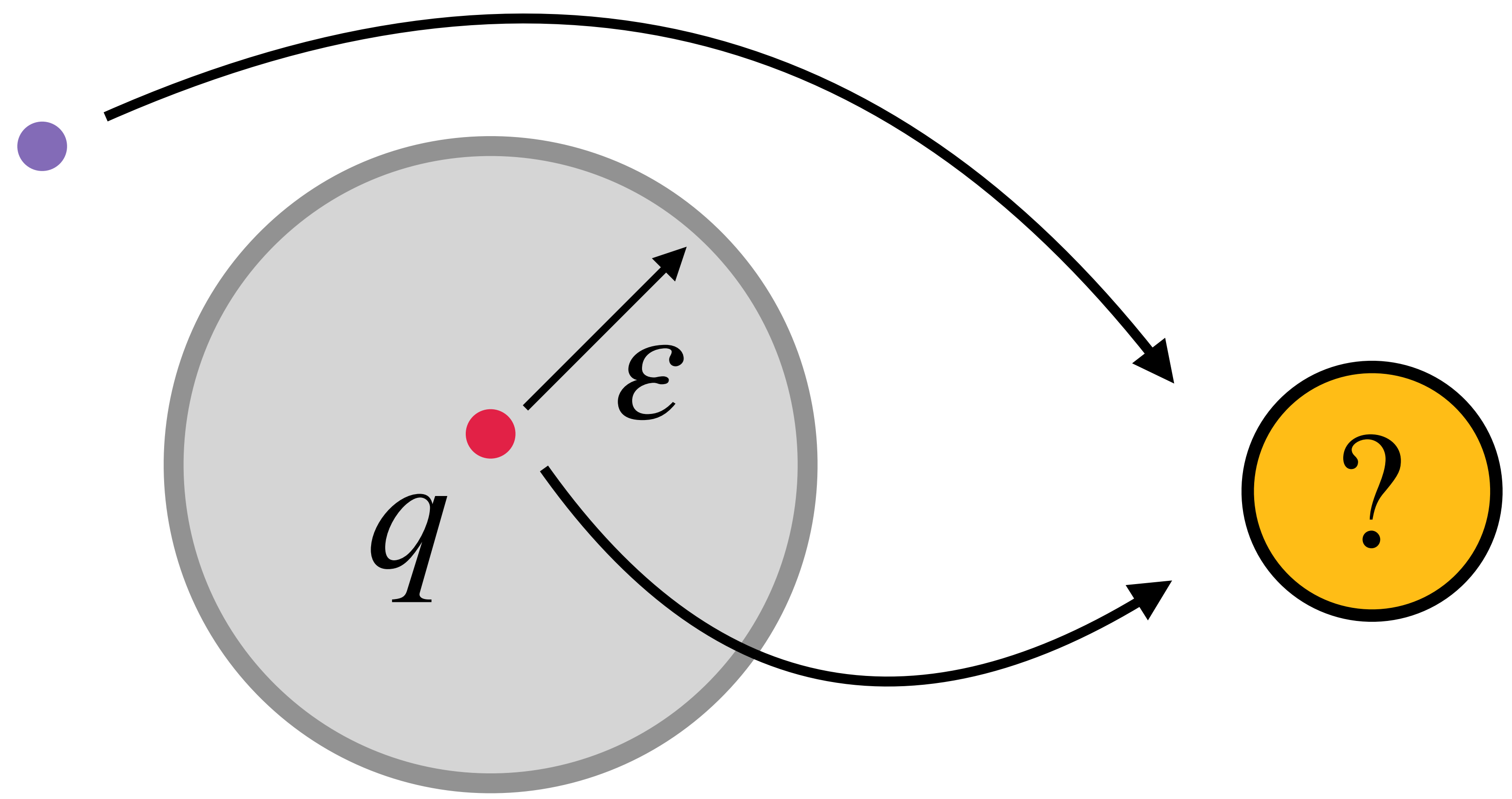

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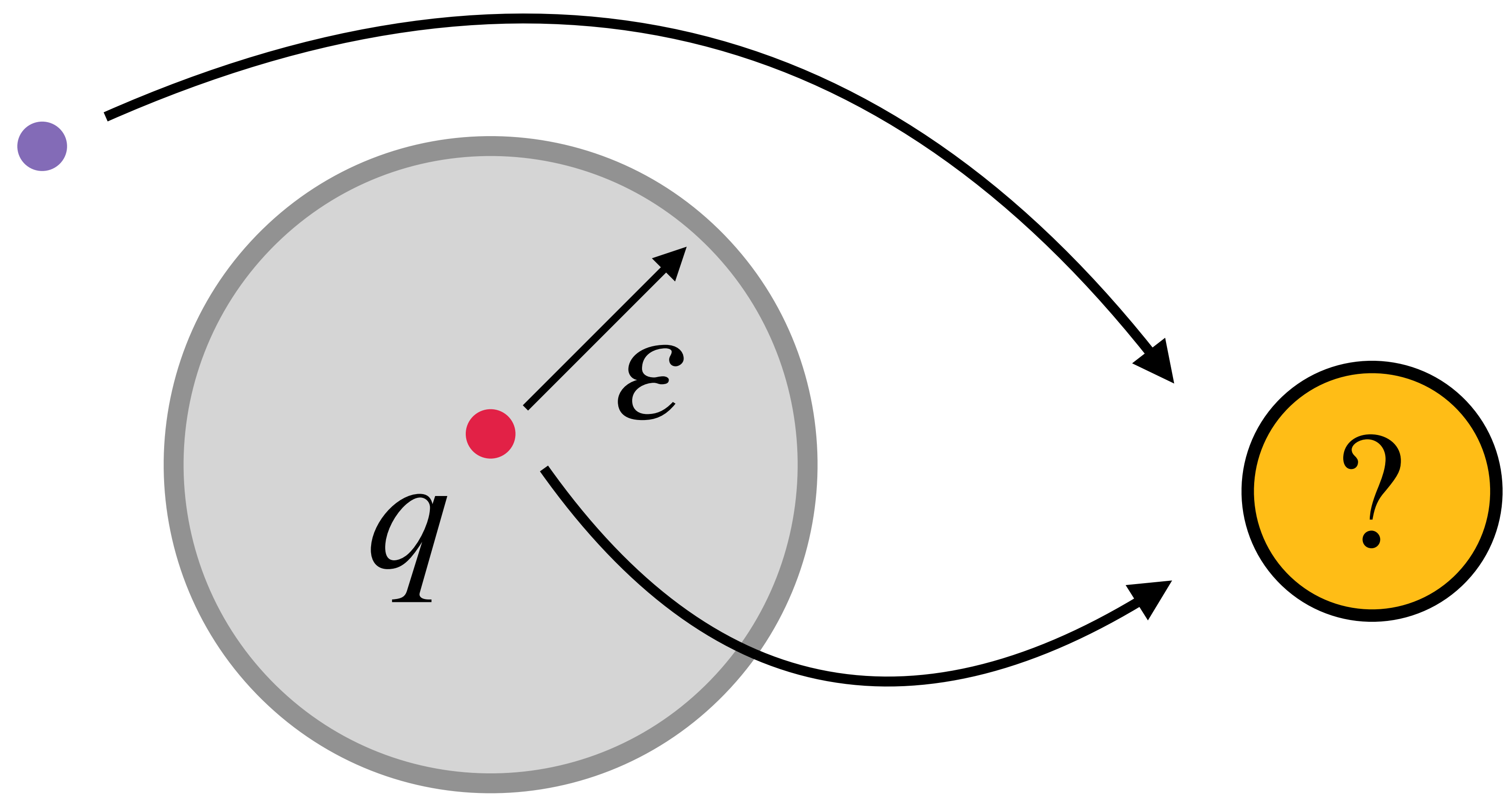

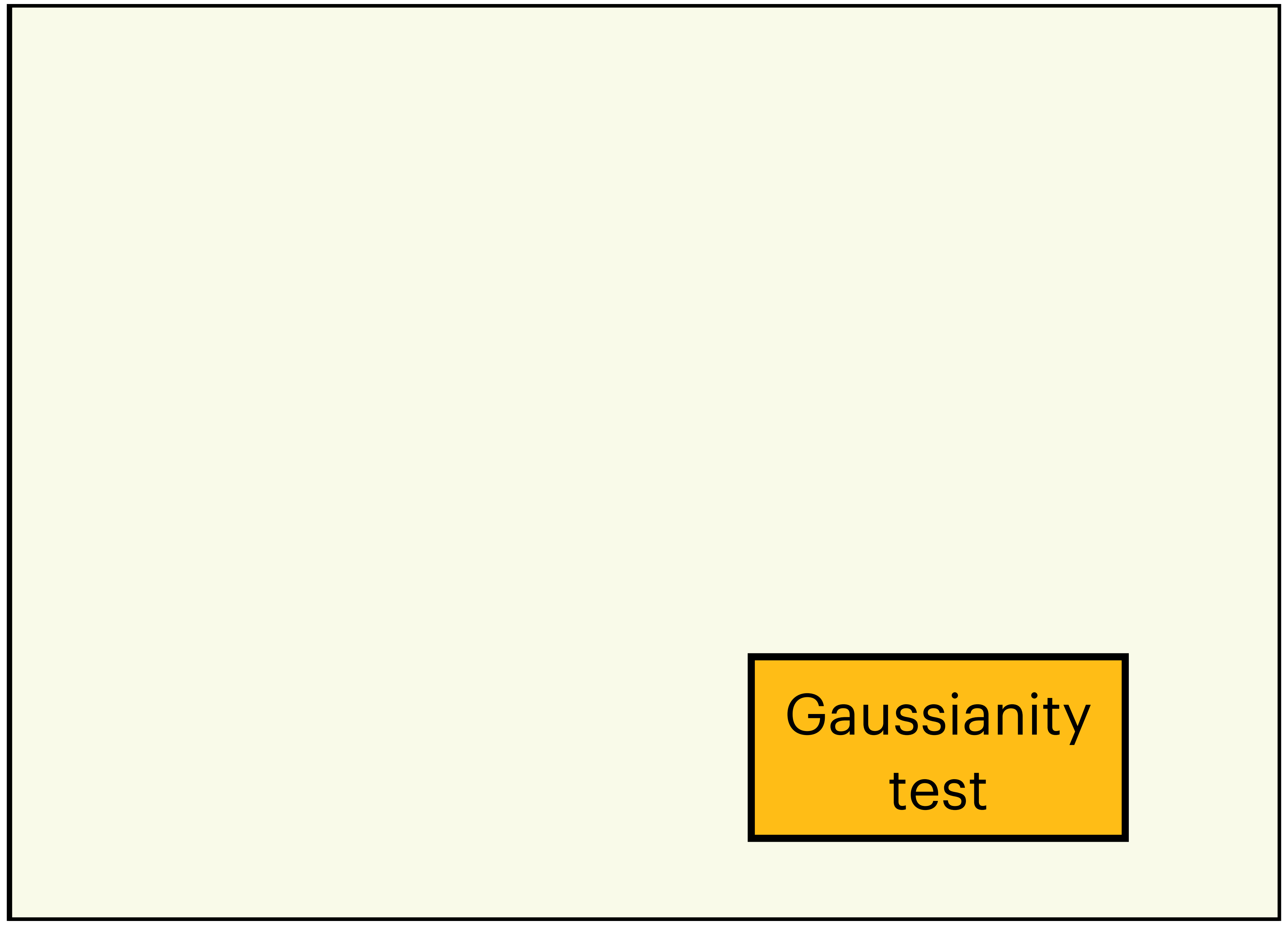

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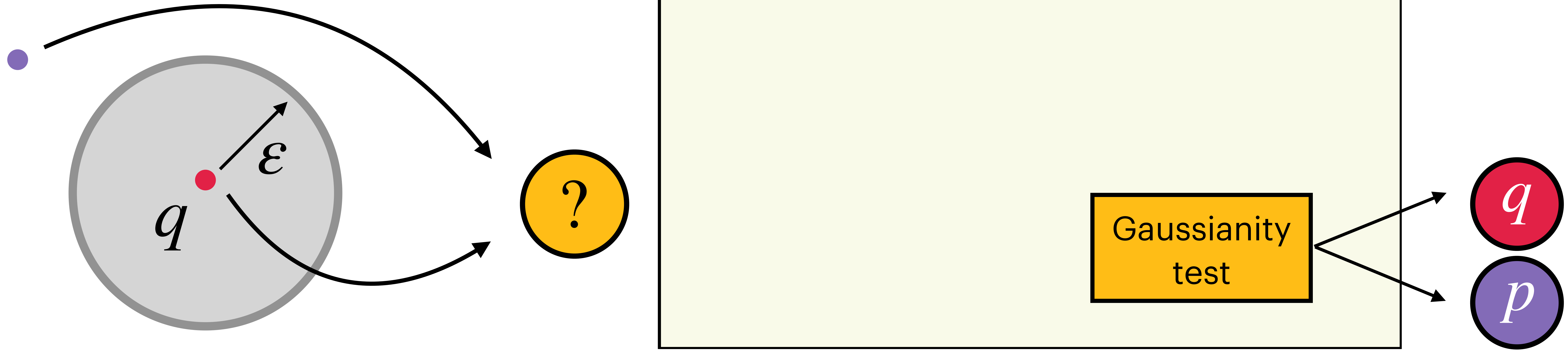

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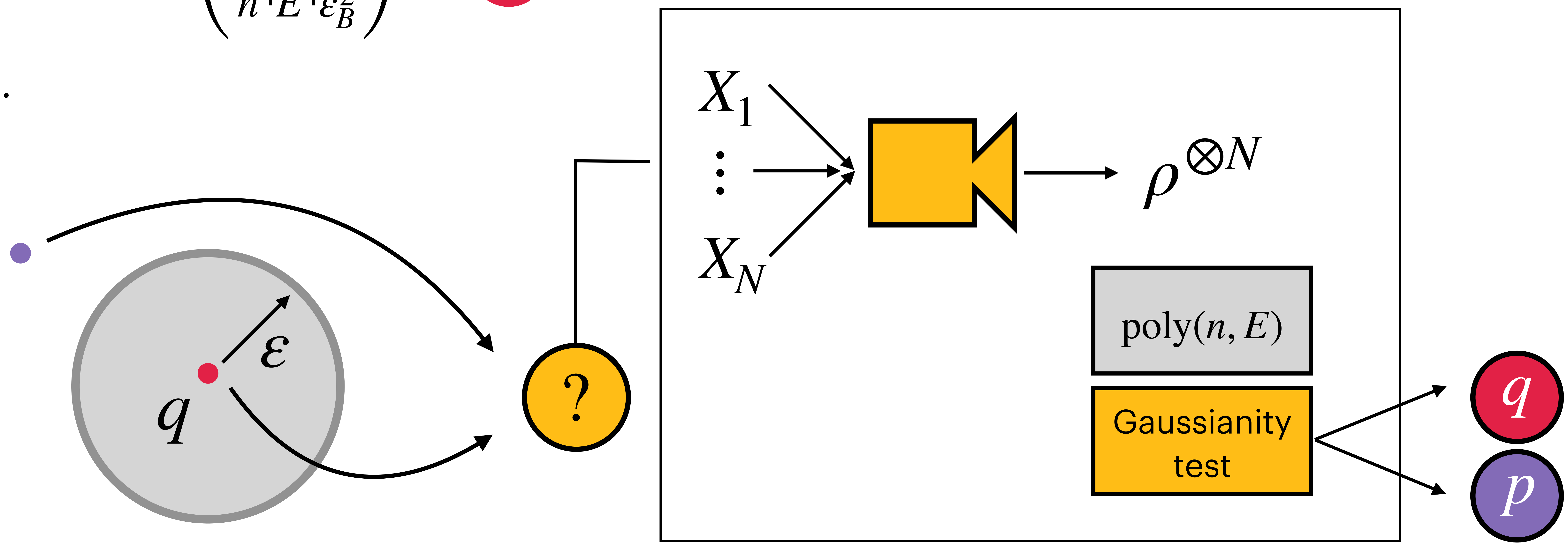
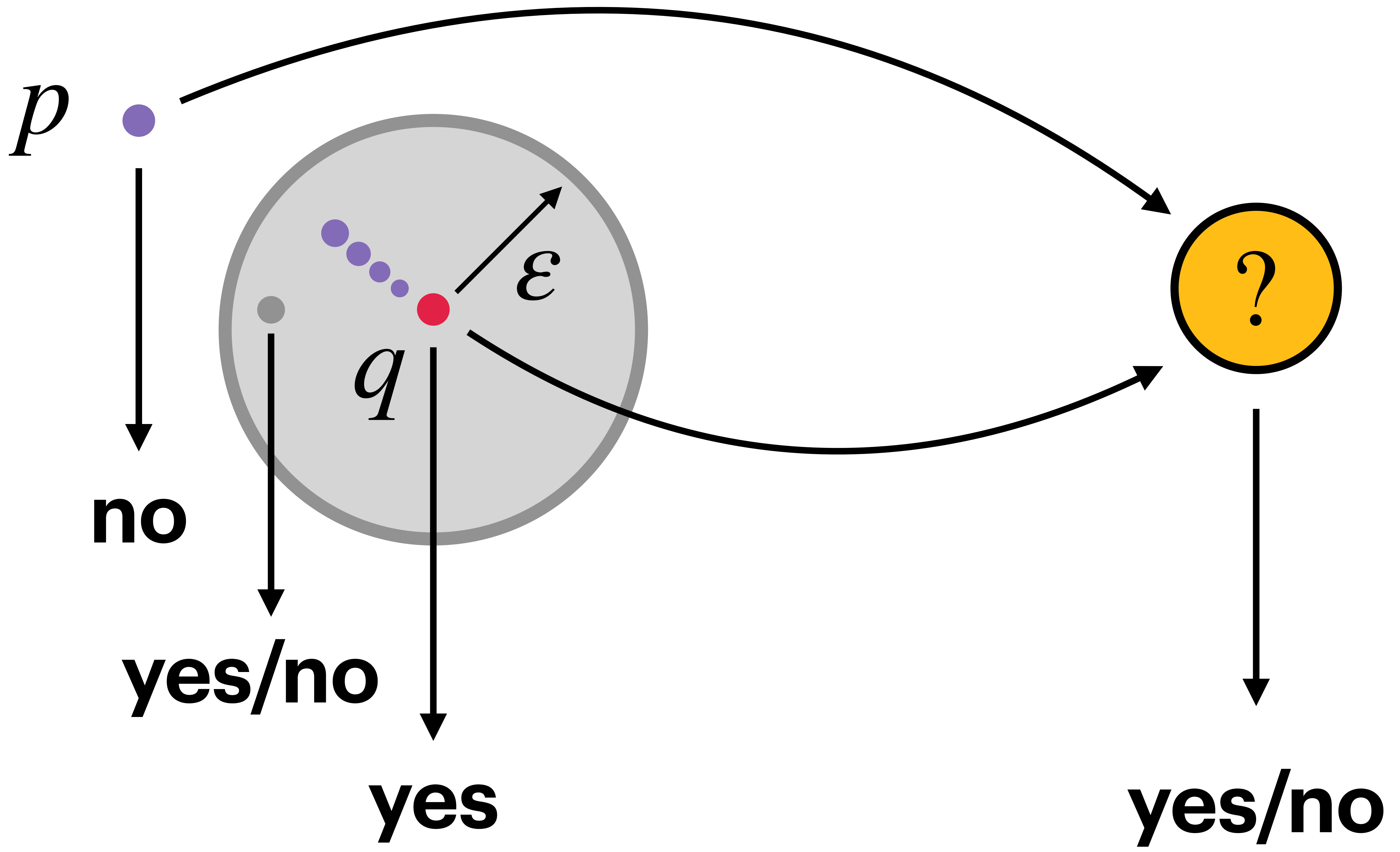


Property testing

The simple case of identity testing



$$X_1, \ldots, X_N \;\; i.i.d.$$

Question: are we observing q

or any other p such that $\frac{1}{2}\|p - q\|_1 \geq \varepsilon$?

$$\boxed{N = \Omega\left(\frac{1}{\varepsilon^2 \|q\|_2}\right) \quad \left(\|q\|_\infty \leq \frac{1}{2}\right)}$$

L. Paninski. IEEE Tr. Inf. Th. 54 (10), 4750-4755 ,
G&P. Valiant, FOCS, 2014,
C. Canonne, Found. Trends Commun. Inf. Theory, Vol. 19 No. 6 pp. 1032–1198

A family of distributions

Proposition. (Classical identity testing) Let q be a probability distribution over $\mathcal{X}$ with $\|q\|_\infty \leq 1/2$ and let $\varepsilon \in (0,1)$. Then there is a family $\mathcal{F}_{q,\varepsilon}$ of probability distributions on $\mathcal{X}$ with the following properties:

- if $p \in \mathcal{F}_{q,\varepsilon}$, then there exists $z \in \{-1, +1\}^{\mathcal{X}}$ such that

$$p(i) = \frac{(1 + 4\varepsilon\, z_i)\, q(i)}{\sum_{j \in \mathcal{X}} (1 + 4\varepsilon\, z_j)\, q(j)} \qquad i \in \mathcal{X}$$

- $\frac{1}{2}\|p - q\|_1 > \varepsilon$ for any $p \in \mathcal{F}_{q,\varepsilon}$;
- testing whether a distribution p is q or belongs to $\mathcal{F}_{q,\varepsilon}$ with failure probability smaller than $1/3$ requires at least $\Omega\left(\frac{1}{\varepsilon^2 \|q\|_2}\right)$ samples of p.

Encoding distributions in quantum states

$$p \quad \rightarrow \quad \rho(p) := \sum_{k \in \mathbb{N}^n} p(k) \, | \, k \rangle \langle k \, |$$

$$| \, k \rangle := | \, k_1 \, k_2 \, \ldots k_n \rangle$$

Encoding distributions in quantum states

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$$q(k) := \frac{1}{(\nu+1)^n}\left(\frac{\nu}{\nu+1}\right)^{\|k\|_1} \quad \to \quad \rho(q) = \tau_\nu^{\otimes n} = \left(\frac{1}{\nu+1}\sum_{k_1=0}^{\infty}\left(\frac{\nu}{\nu+1}\right)^{k_1}|k_1\rangle\langle k_1|\right)^{\otimes n}$$

This state is Gaussian!

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This state is Gaussian!

$$\mathrm{Tr}[\tau_\nu^{\otimes n} \hat{E}^2] = n\nu(2\nu + 1) + n(n - 1)\nu^2 + n^2\nu + \frac{n^2}{4} \equiv \frac{1 - 4\varepsilon}{1 + 4\varepsilon} n^2 E^2 \quad \rightarrow \quad \nu = \Omega(E)$$

Encoding distributions in quantum states

$$\mathrm{Tr}[\rho(q)\hat{E}^2] = \frac{1-4\varepsilon}{1+4\varepsilon} n^2 E^2 \leq n^2 E^2$$

Encoding distributions in quantum states

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$$\frac{1}{2}\|\rho(p) - \rho(q)\|_1 = \frac{1}{2}\|p - q\|_1 > \varepsilon$$

Encoding distributions in quantum states

$$\boxed{\begin{aligned}&\mathrm{Tr}[\rho(q)\hat{E}^2] \le n^2E^2\\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \le n^2E^2\\ &\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon\end{aligned}}$$

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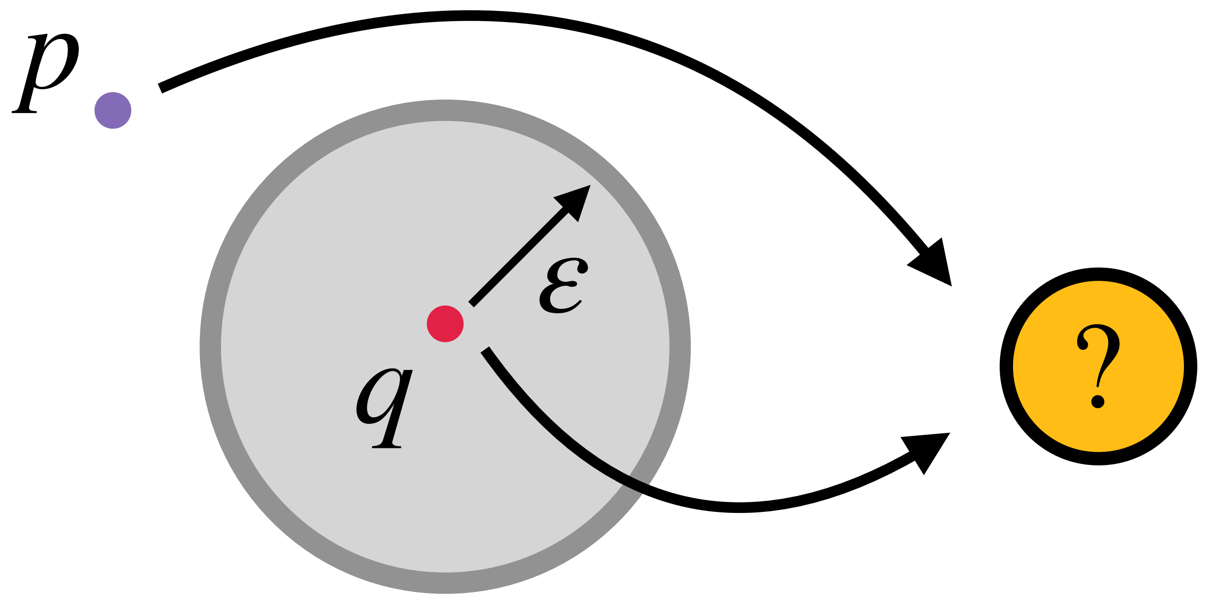
Encoding distributions in quantum states



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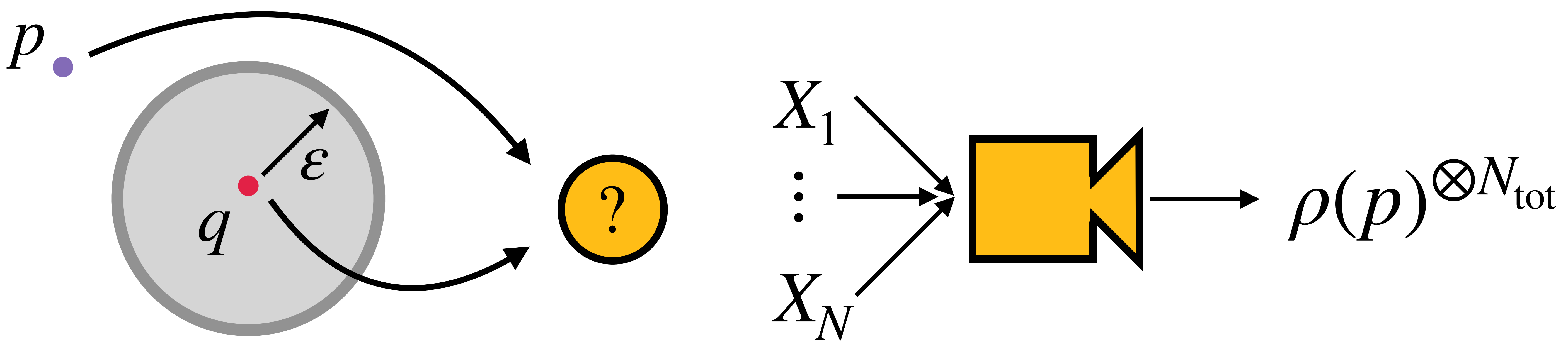

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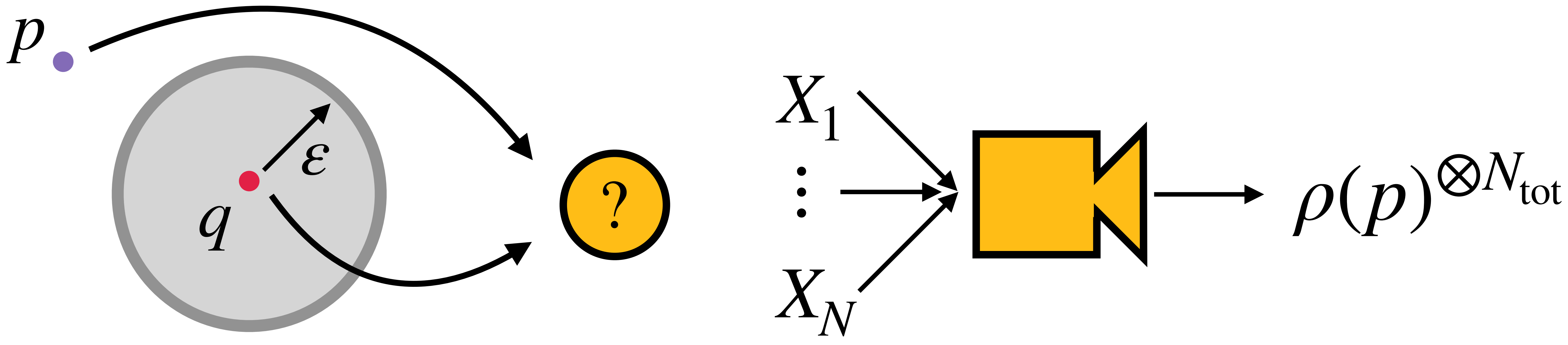

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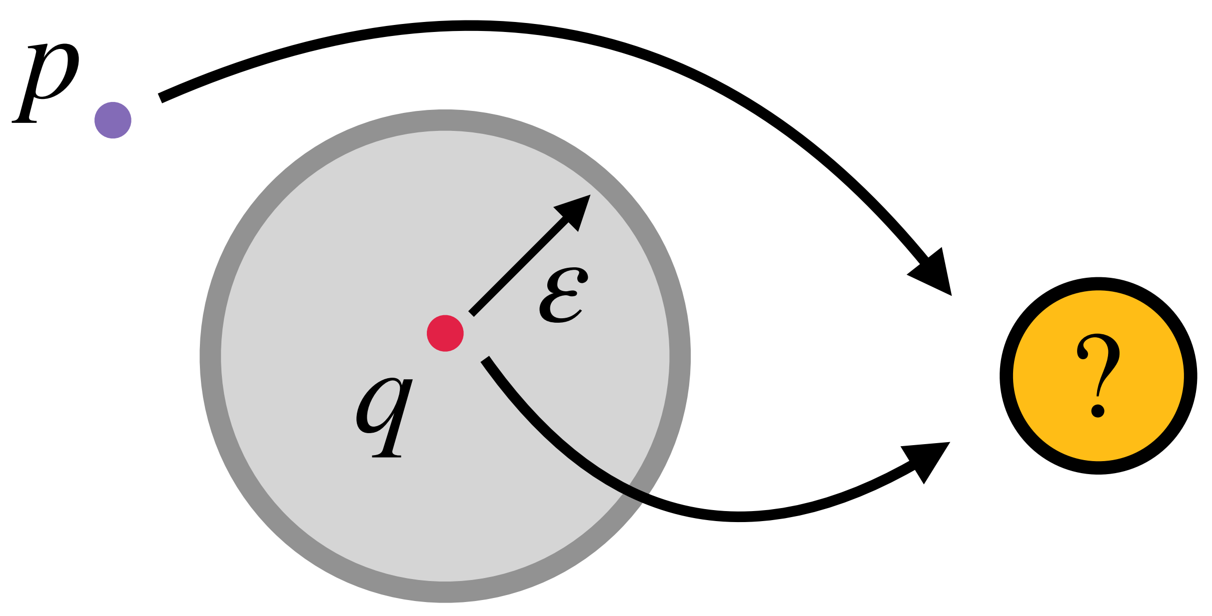
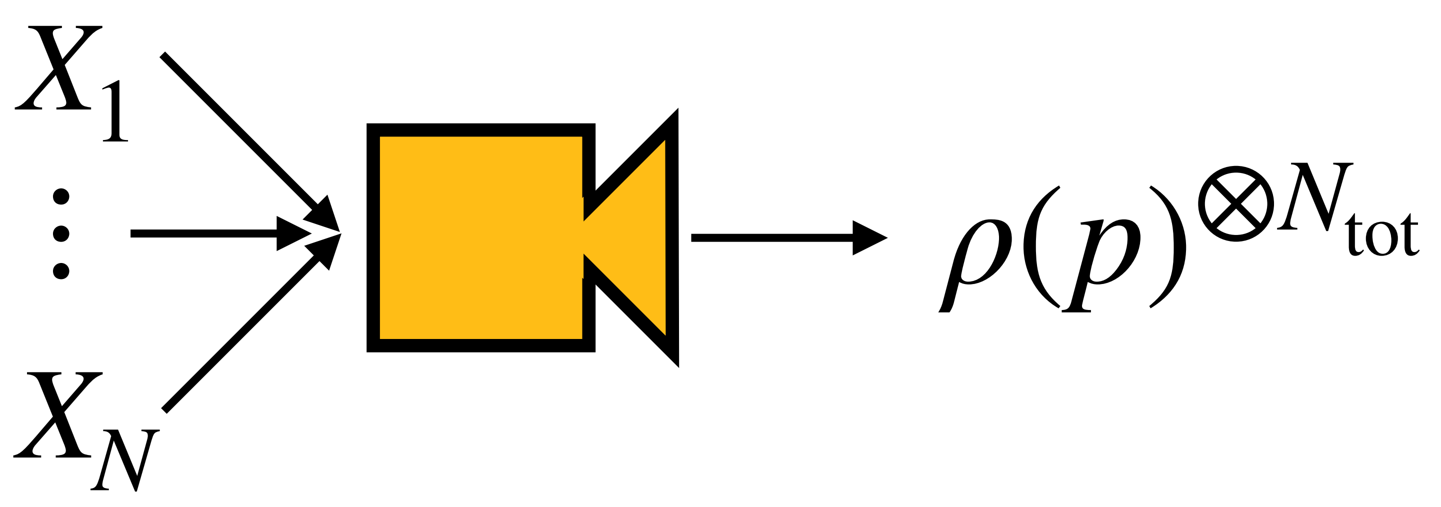
If we sample $\{X_i = x_i\}_{i=1,\ldots,N_{\mathrm{tot}}}$, then we prepare $|x_1\rangle\langle x_1| \otimes \cdots \otimes |x_{N_{\mathrm{tot}}}\rangle\langle x_{N_{\mathrm{tot}}}|$.
The state we obtain is exactly

$$\sum_{x_1,\ldots,x_{N_{\mathrm{tot}}}\in\mathbb{N}^n} p(x_1)\cdots p(x_{N_{\mathrm{tot}}})\,|x_1\rangle\langle x_1| \otimes \cdots \otimes |x_{N_{\mathrm{tot}}}\rangle\langle x_{N_{\mathrm{tot}}}| = \left(\sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|\right)^{\otimes N_{\mathrm{tot}}} = \rho(p)^{\otimes N_{\mathrm{tot}}}$$

Encoding distributions in quantum states





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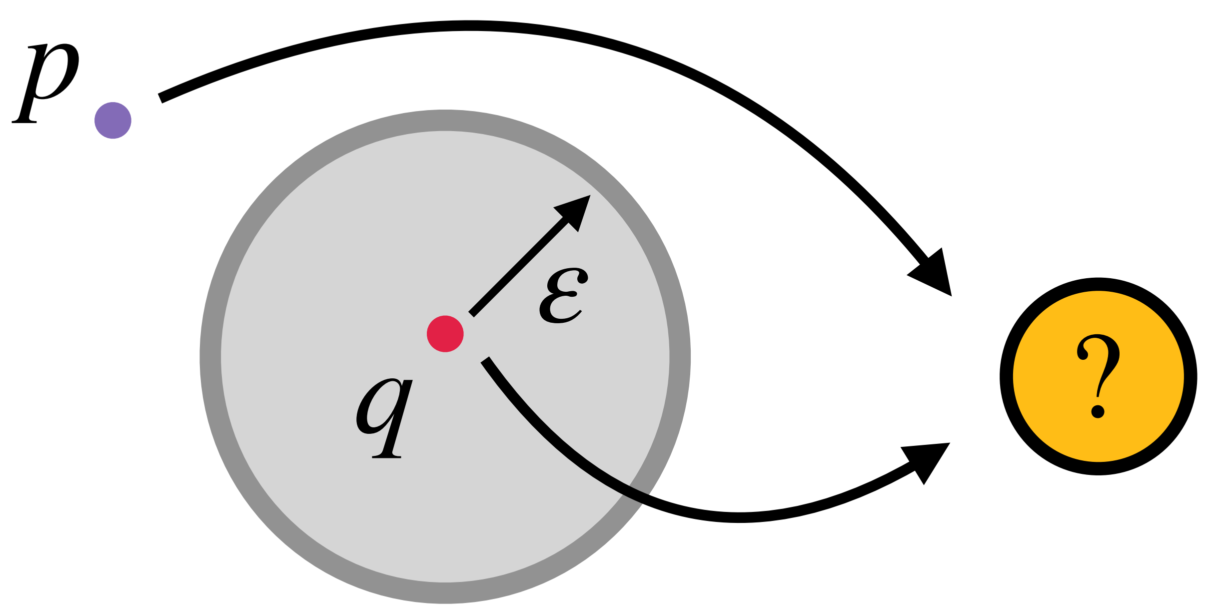
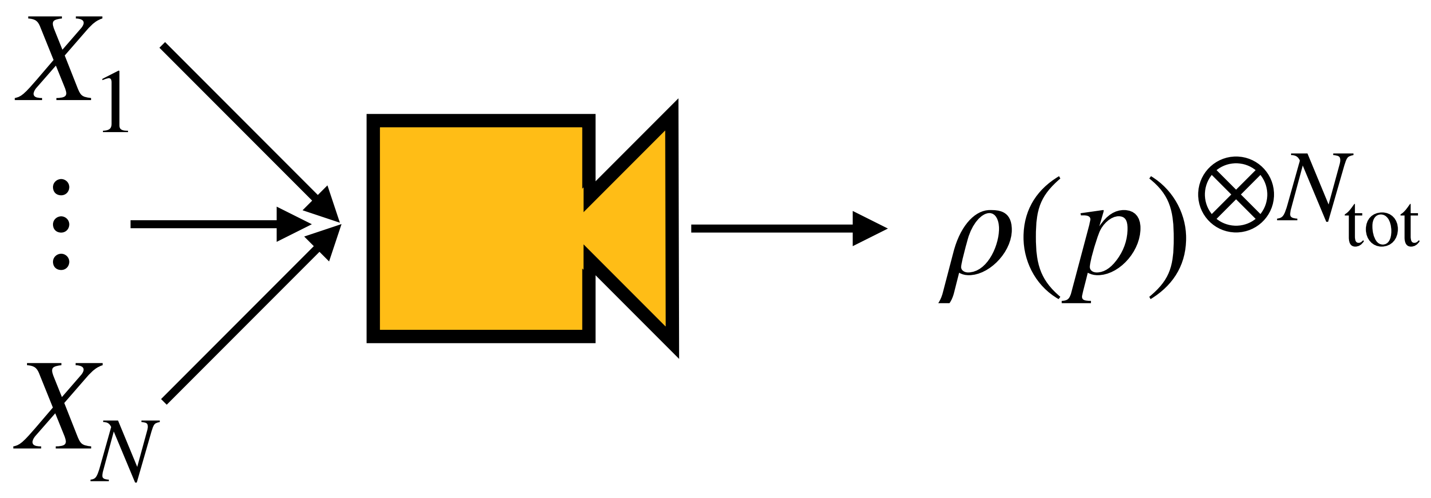
Encoding distributions in quantum states





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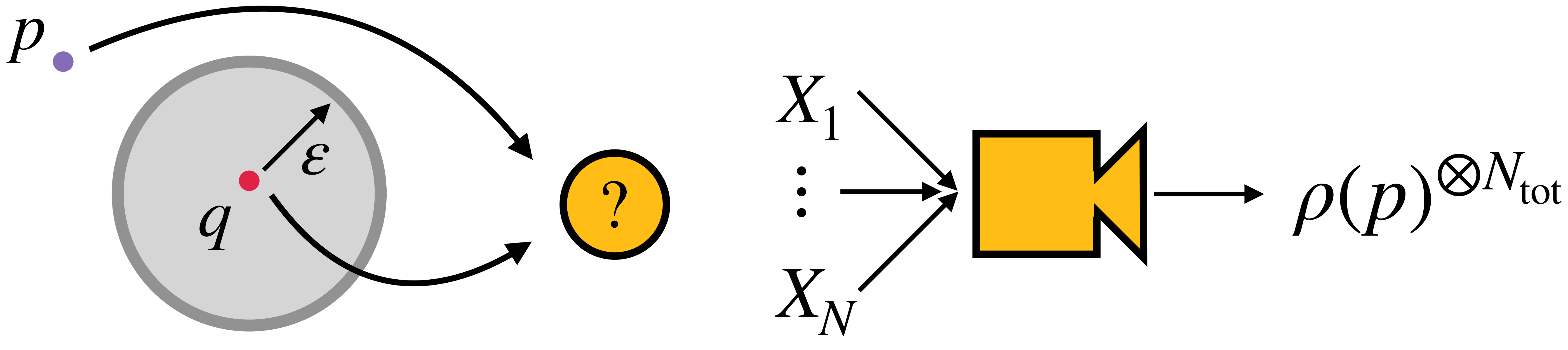
Step 1: testing the covariance matrix



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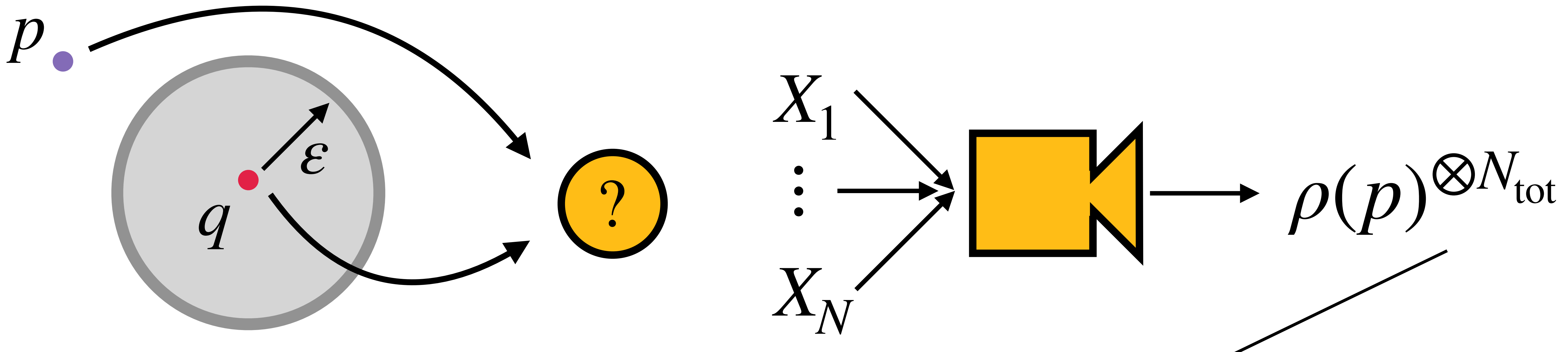

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Mele & al. Nature Physics 21, 2002-2008 (2025)

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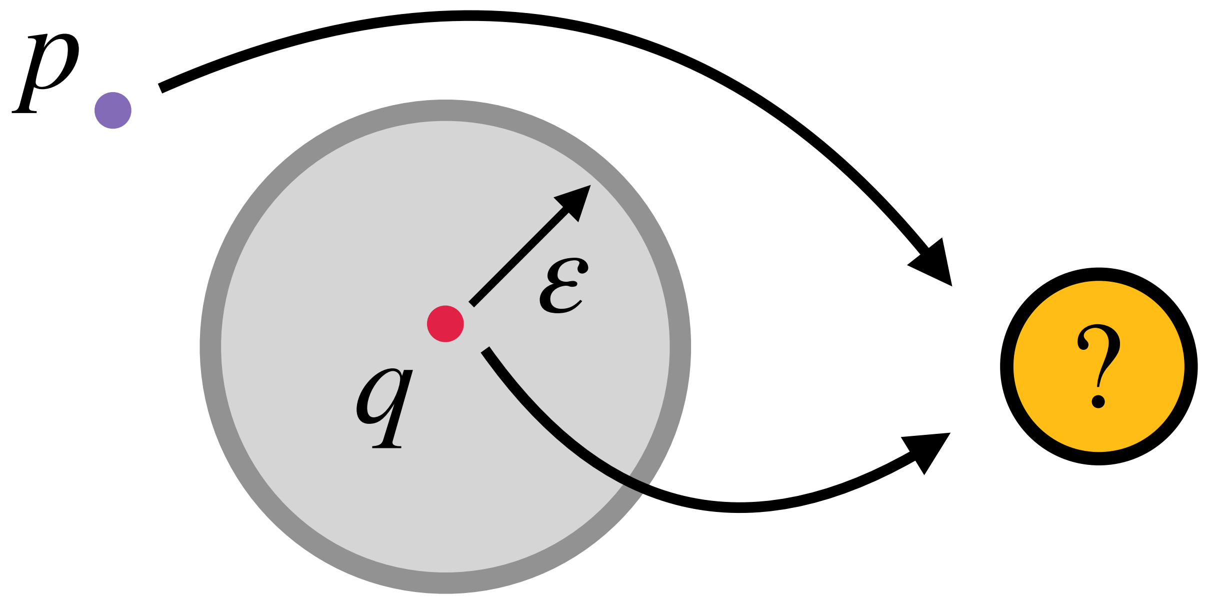

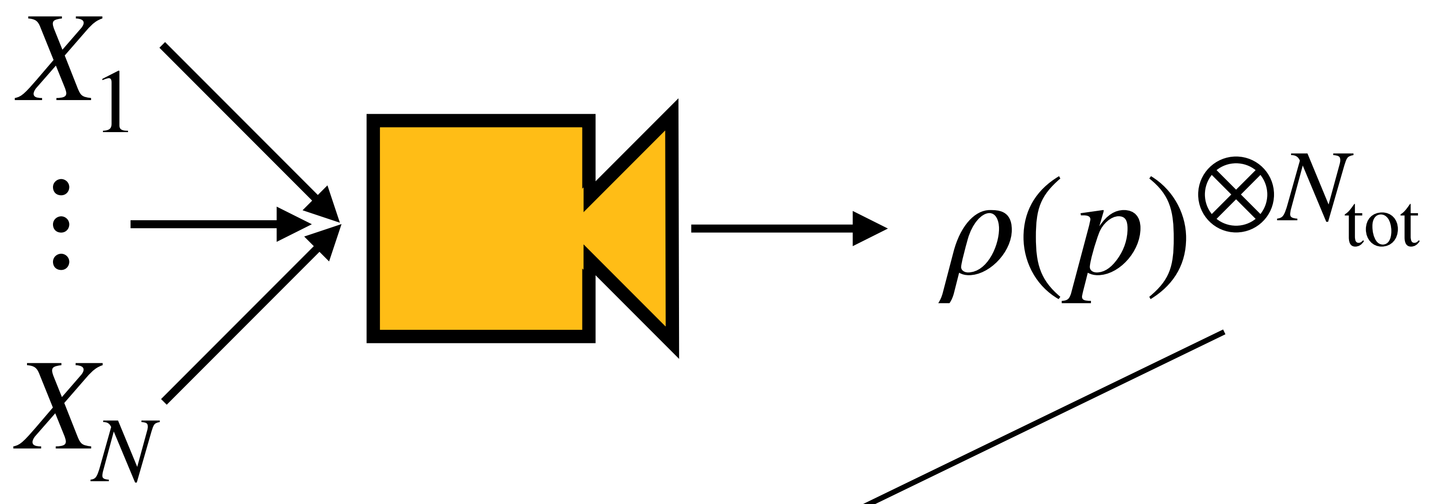

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Mele & al. Nature Physics 21, 2002-2008 (2025)

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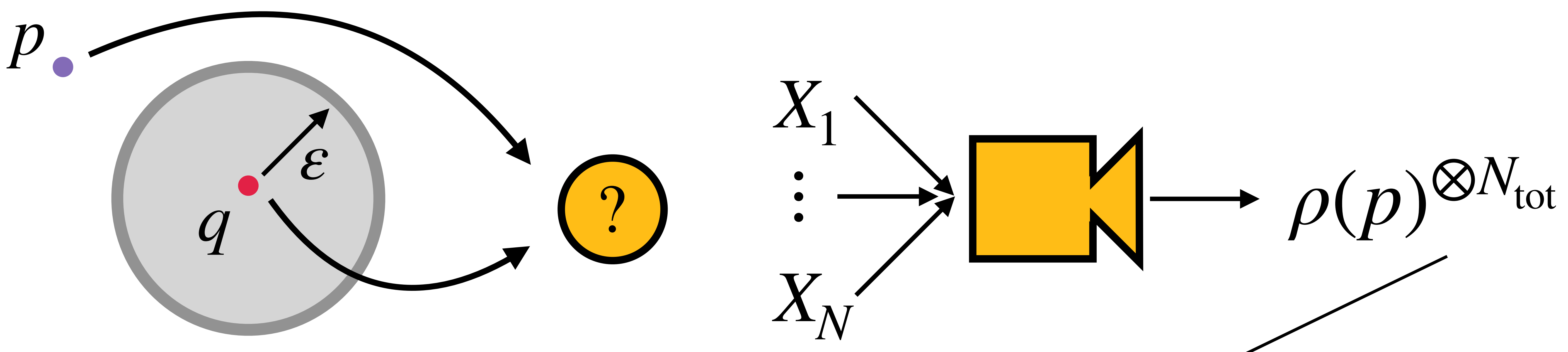

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Mele & al. Nature Physics 21, 2002-2008 (2025)

$\rho(p)^{\otimes N_{\text{cov}}} \longrightarrow$  $= \tilde{V}(p)$

$$\|\tilde{V}(p) - V(p)\|_2 \leq \varepsilon_V$$

$$\varepsilon_V = \frac{\varepsilon}{4(1+\sqrt{3})nE}$$

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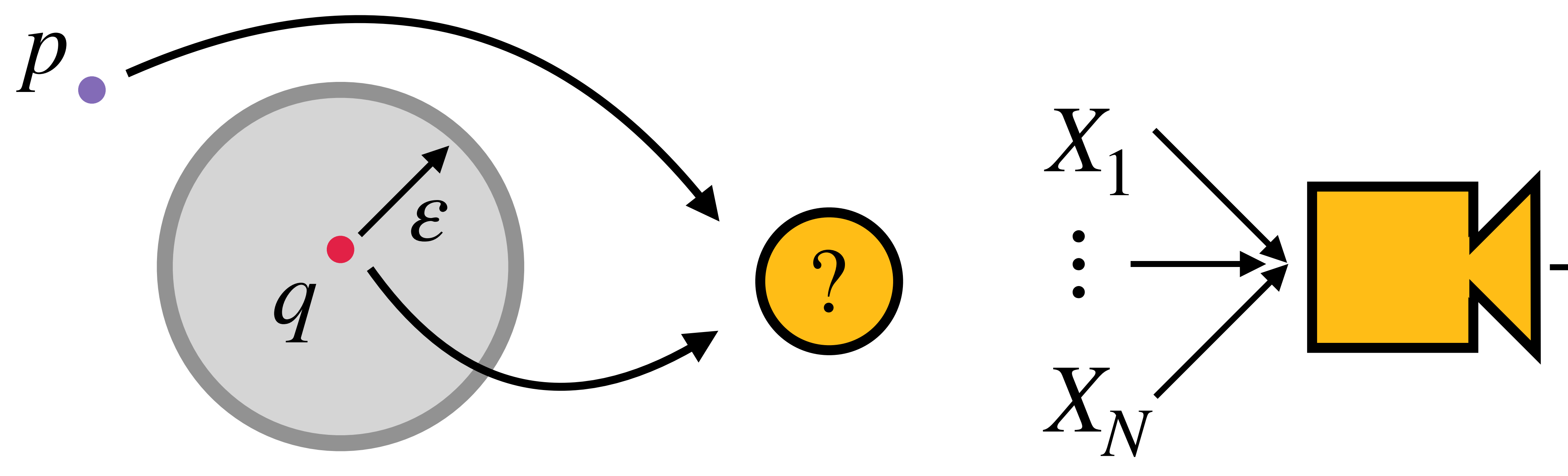

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$$\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon$$

$$N_{\text{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Mele & al. Nature Physics 21, 2002-2008 (2025)

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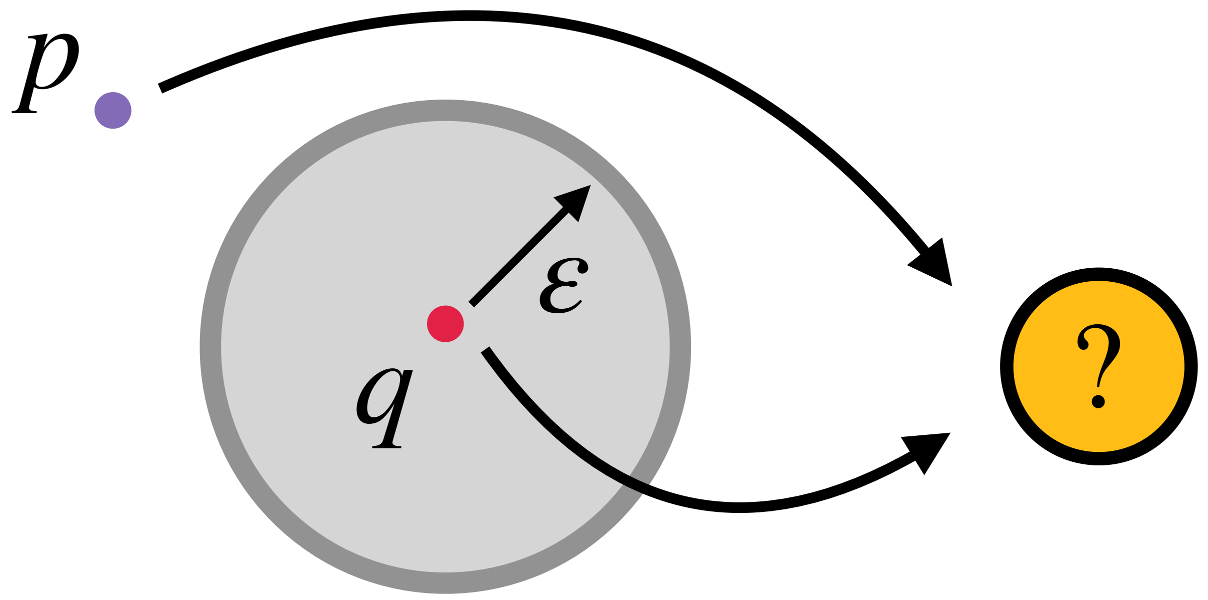

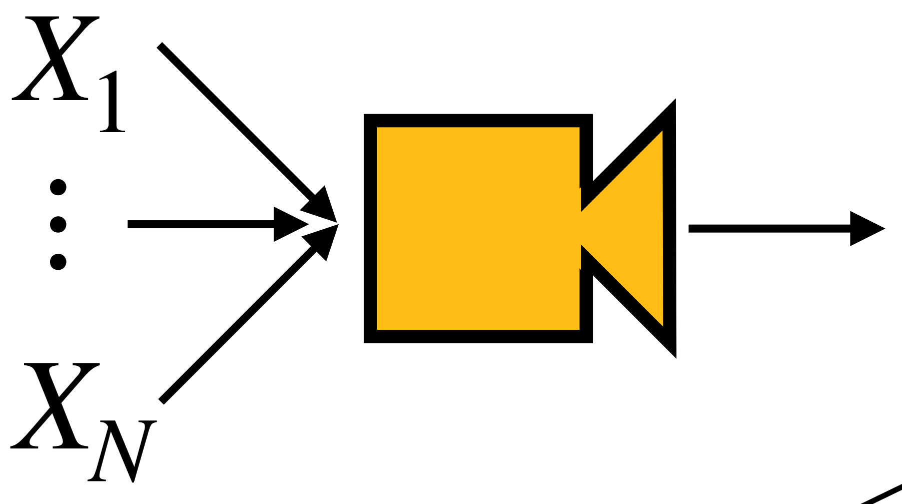

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$$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon$$

$$N_{\text{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Mele & al. Nature Physics 21, 2002-2008 (2025)

$\rho(p)^{\otimes N_{\text{cov}}} \longrightarrow$  $= \tilde{V}(p)$

$$\|\tilde{V}(p)-V(p)\|_2 \leq \varepsilon_V$$
$$\varepsilon_V = \frac{\varepsilon}{4(1+\sqrt{3})nE}$$

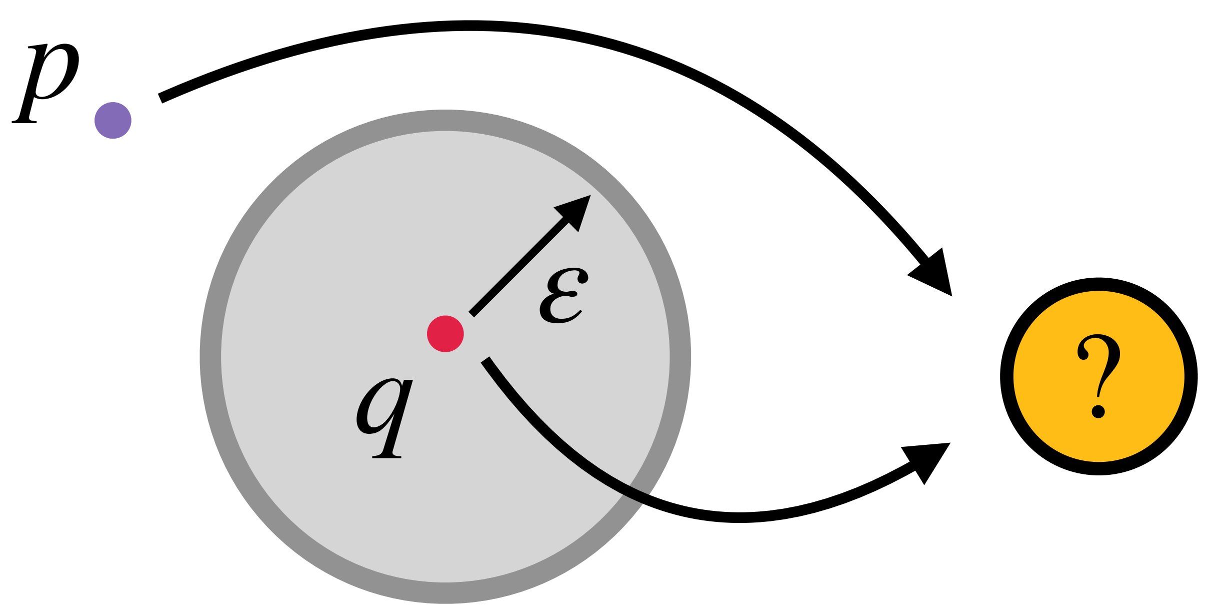
Now, if $\|\tilde{V}(p)-V(q)\|_2 > \varepsilon_V$,

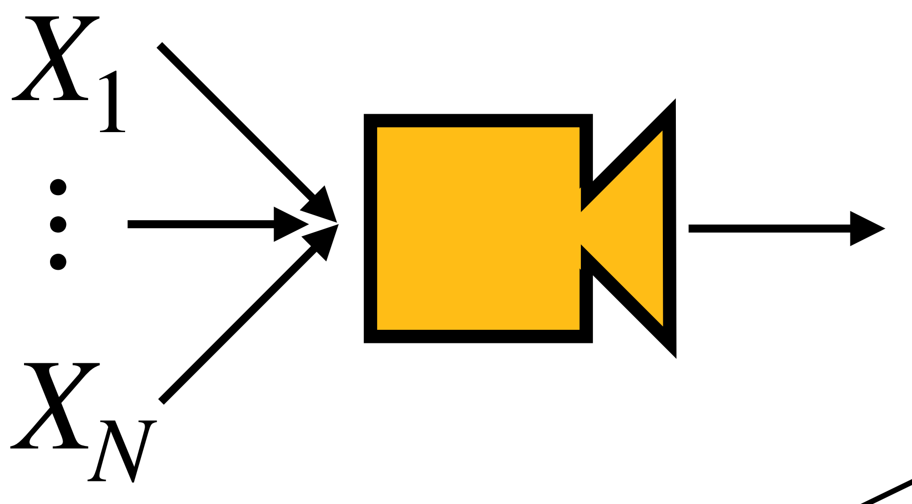
then $\|V(p)-V(q)\|_2 \geq \|\tilde{V}(p)-V(q)\|_2 - \|\tilde{V}(p)-V(p)\|_2 > 0$,

so we output "**not** q", since

$\|p-q\|_1 = \|\rho(p)-\rho(q)\|_1 > 0$

Step 1: testing the covariance matrix





$\rho(p)^{\otimes N_{\mathrm{tot}}}$

$$\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2$$
$$\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2$$
$$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon$$

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Mele & al. Nature Physics 21, 2002-2008 (2025)

$\rho(p)^{\otimes N_{\mathrm{cov}}} \longrightarrow$  $= \tilde{V}(p)$

$$\|\tilde{V}(p)-V(p)\|_2 \leq \varepsilon_V$$
$$\varepsilon_V = \frac{\varepsilon}{4(1+\sqrt{3})nE}$$

Now, if $\|\tilde{V}(p)-V(q)\|_2 > \varepsilon_V$,

then $\|V(p)-V(q)\|_2 \geq \|\tilde{V}(p)-V(q)\|_2 - \|\tilde{V}(p)-V(p)\|_2 > 0$,

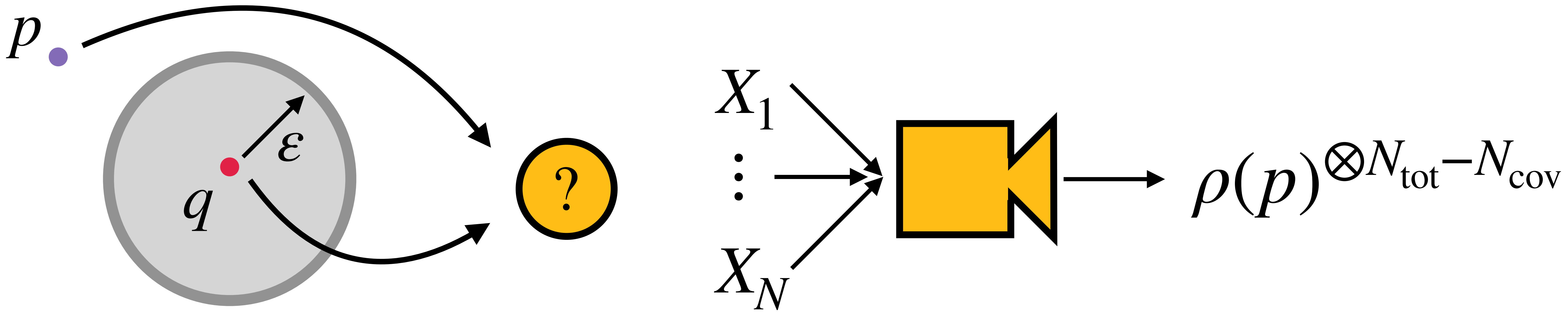
so we output "**not** q", since

$\|p-q\|_1 = \|\rho(p)-\rho(q)\|_1 > 0$

Lower bound between arbitrary states.

$$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 \geq \frac{\|V(p)-V(q)\|_\infty^2}{3098\ \max(\mathrm{Tr}[\hat{E}^2\rho(p)],\mathrm{Tr}[\hat{E}^2\rho(q)])}$$

Step 2: Gaussianity test



$\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2$

$\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2$

$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon$

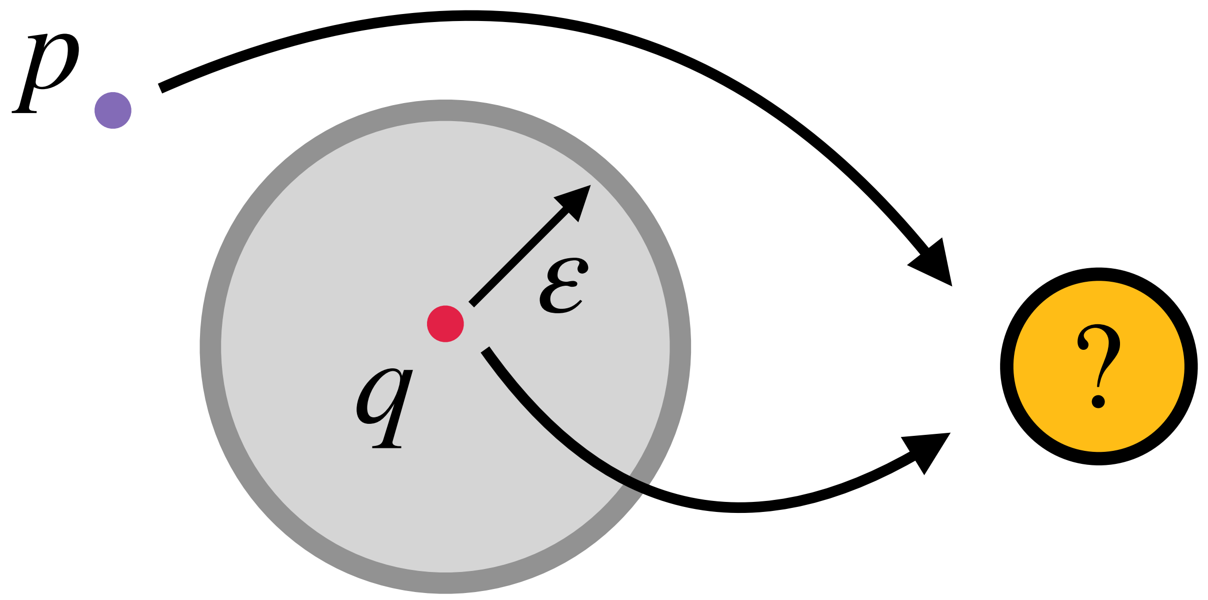
$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

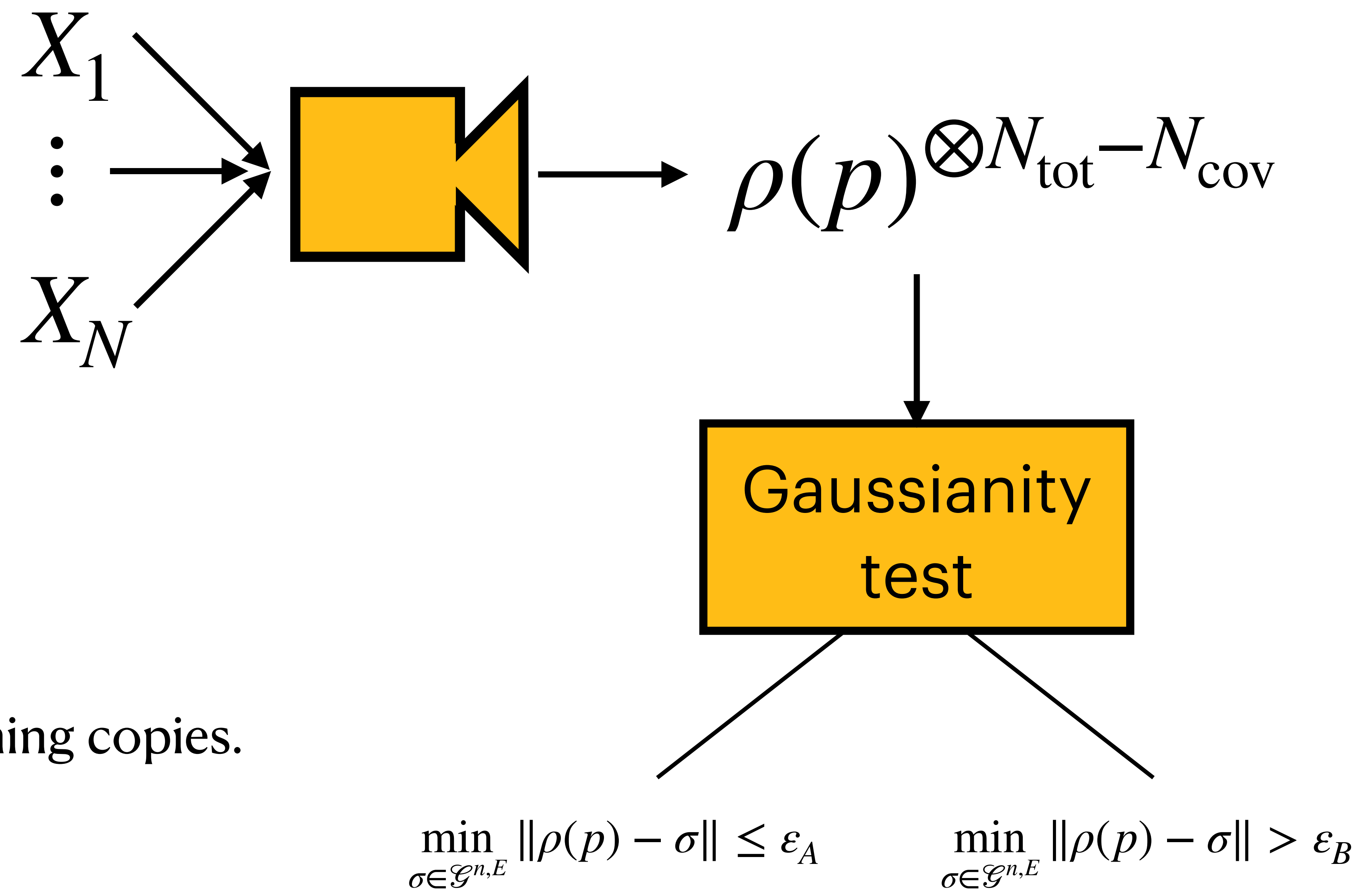
Otherwise, if $\|\tilde{V}(p)-V(q)\|_2 \leq \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

$$N_{\rm cov} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

Step 2: Gaussianity test





$$\boxed{\begin{aligned} &\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2 E^2 \\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2 E^2 \\ &\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon \end{aligned}}$$

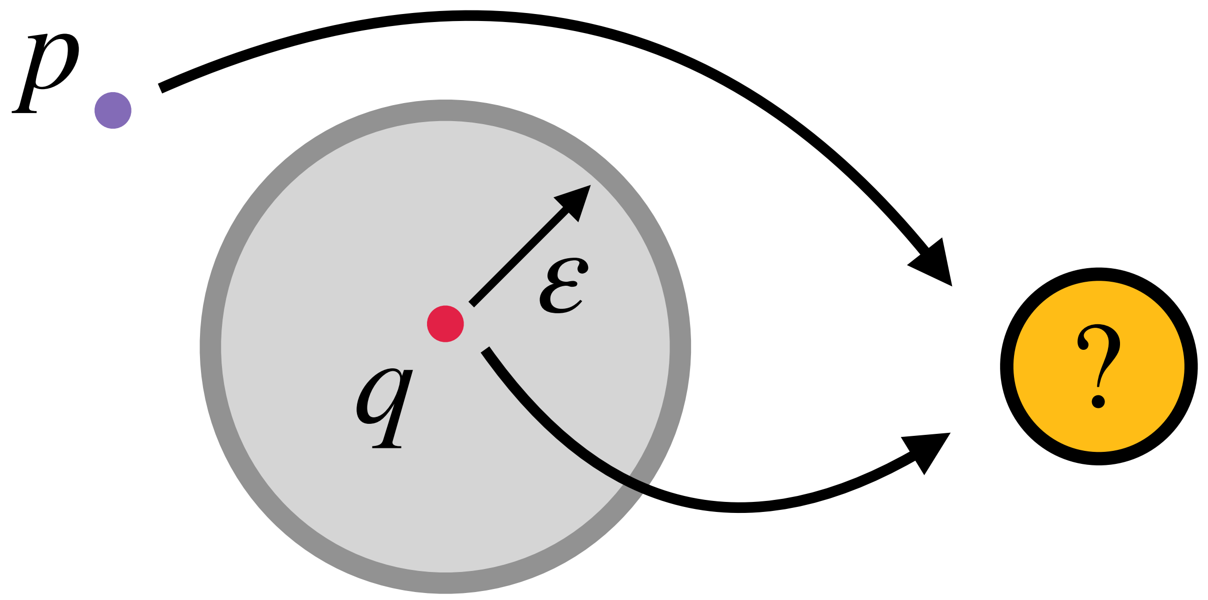
$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

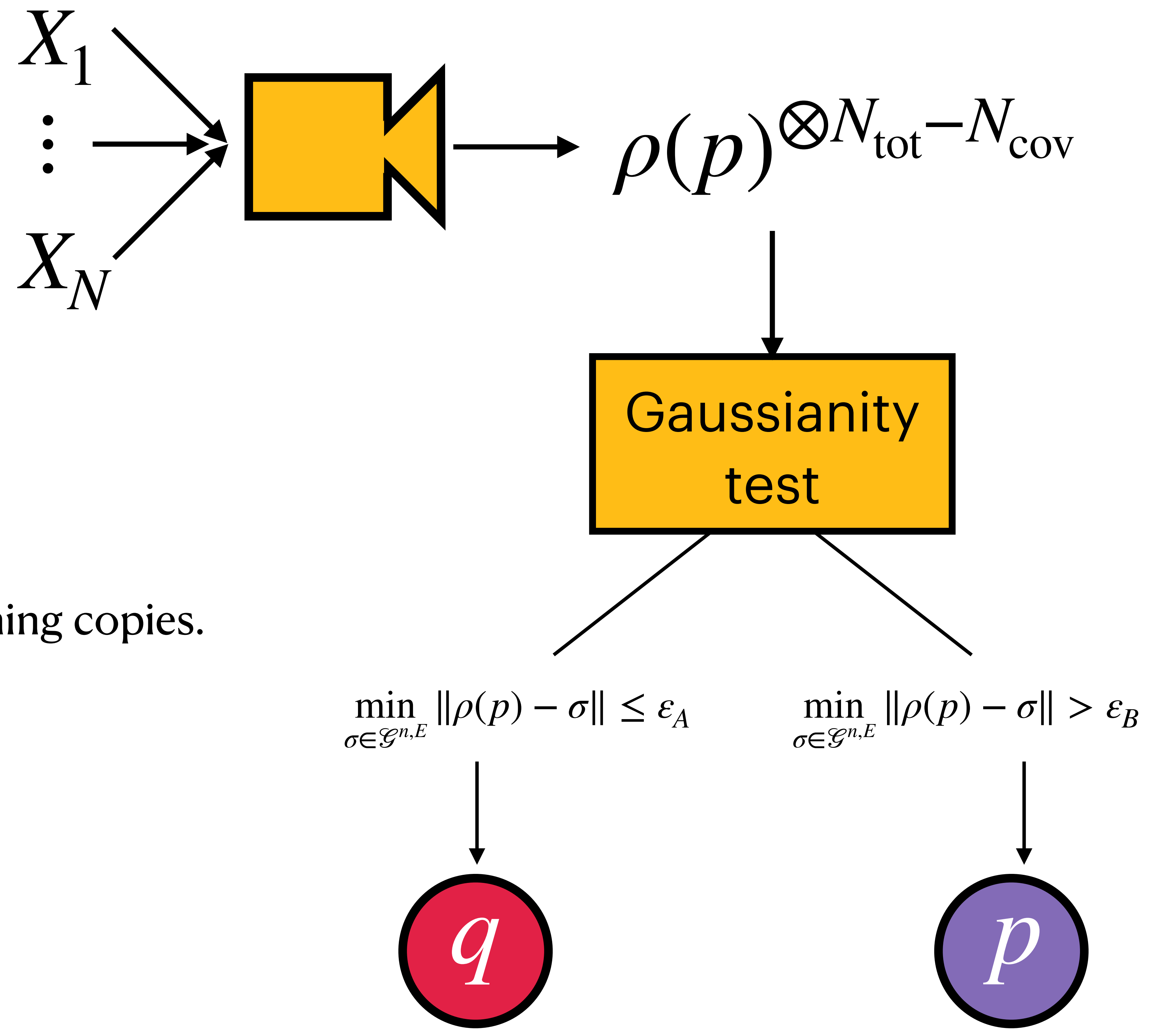
$$N_{\text{cov}} = O\left(\log\left(n^2\right) \frac{n^5 E^4}{\varepsilon^2}\right)$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

Step 2: Gaussianity test





$$\boxed{\begin{array}{l} \mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2 \\ \mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2 \\ \frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon \end{array}}$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

$$N_{\text{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

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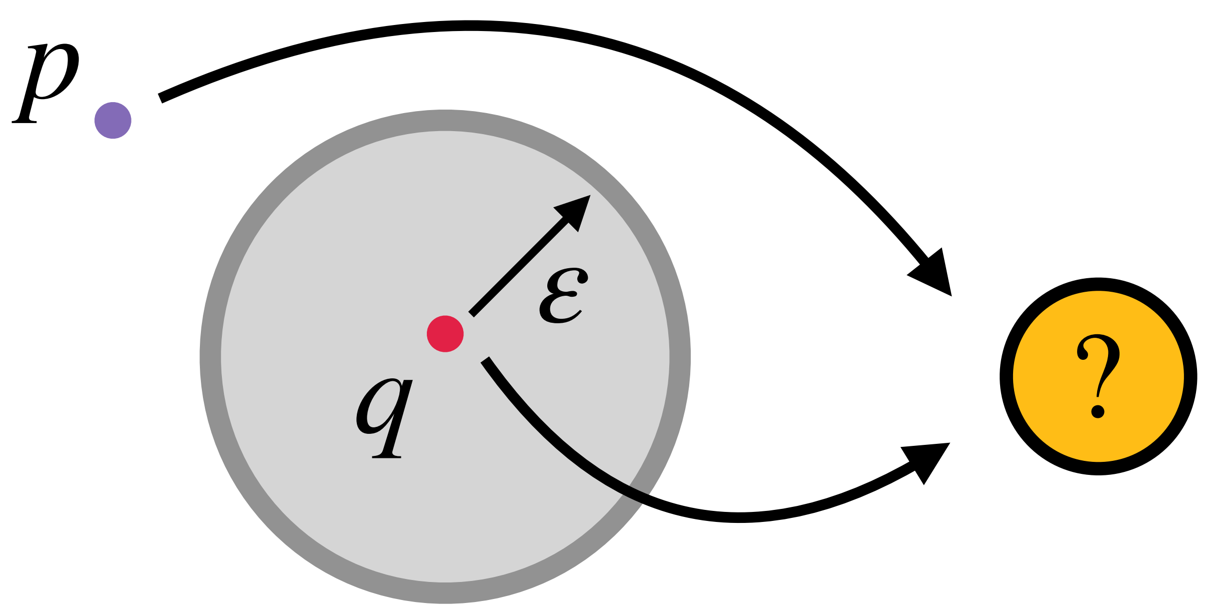
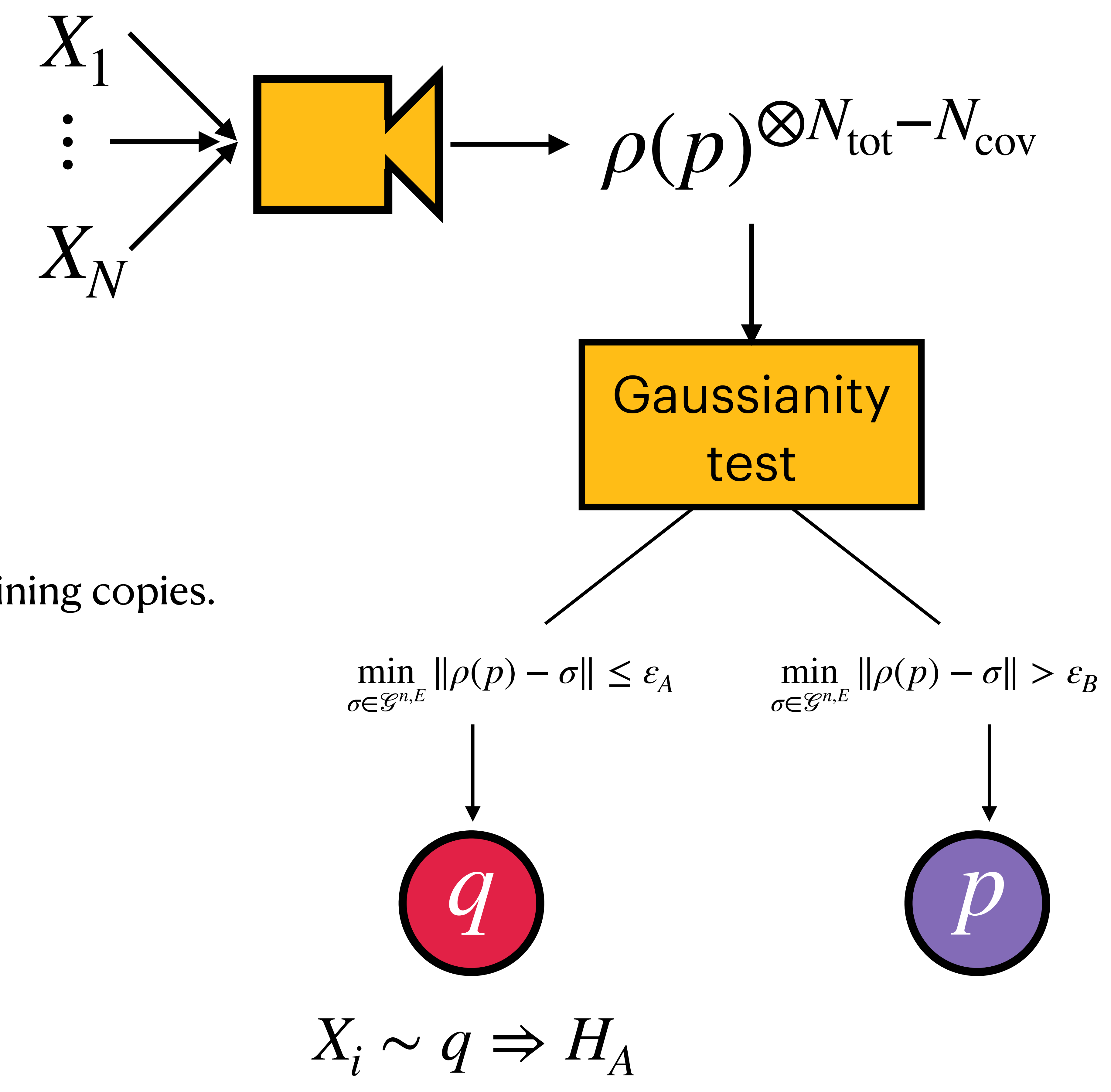
Step 2: Gaussianity test





$$\boxed{\begin{aligned} &\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2 \\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2 \\ &\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon \end{aligned}}$$

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Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

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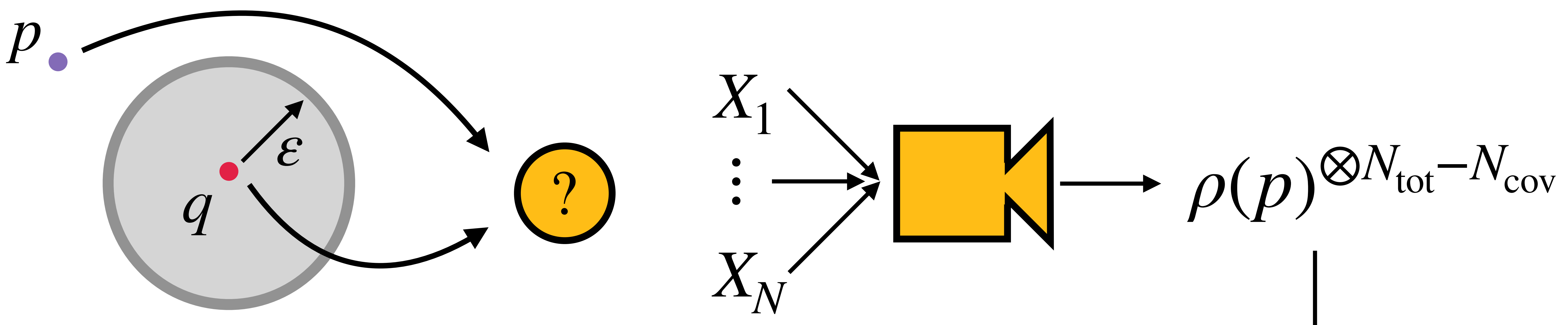
Step 2: Gaussianity test



$$\mathrm{Tr}[\rho(q)\hat{E}^2] \le n^2E^2$$

$$\mathrm{Tr}[\rho(p)\hat{E}^2] \le n^2E^2$$

$$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \le \varepsilon_V$,

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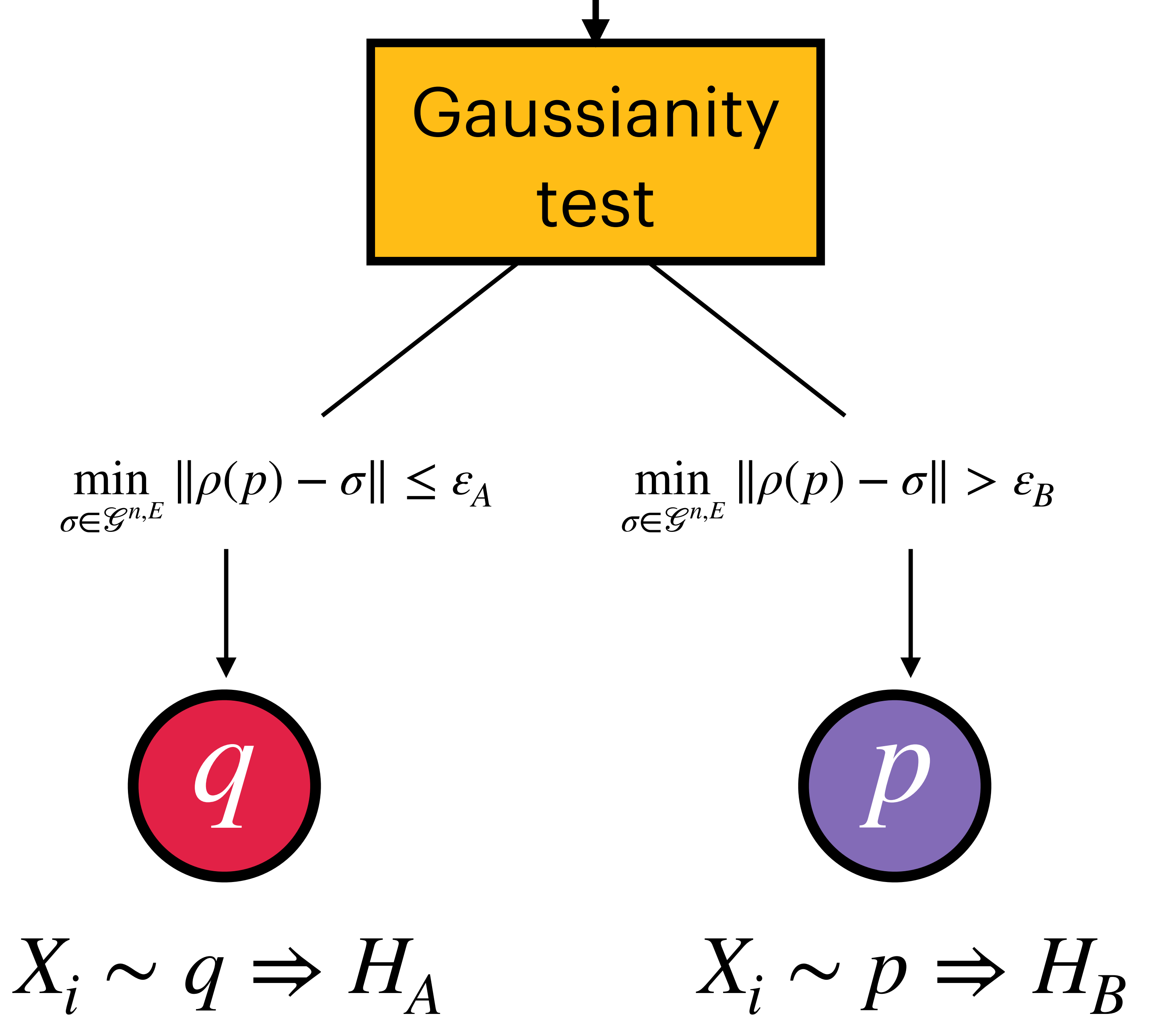
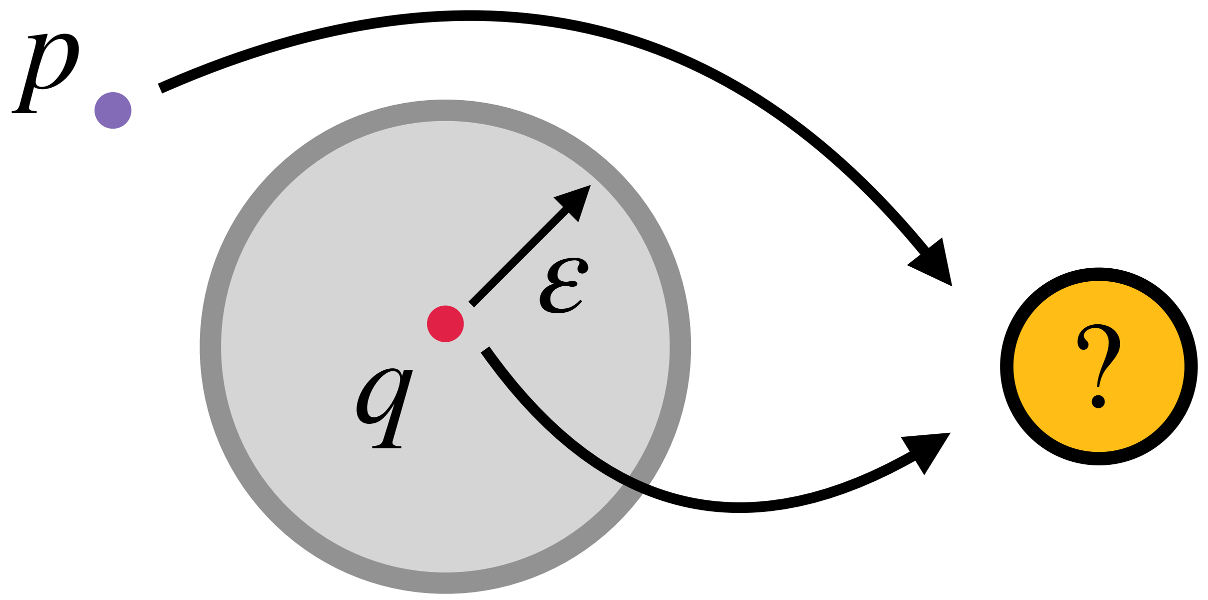
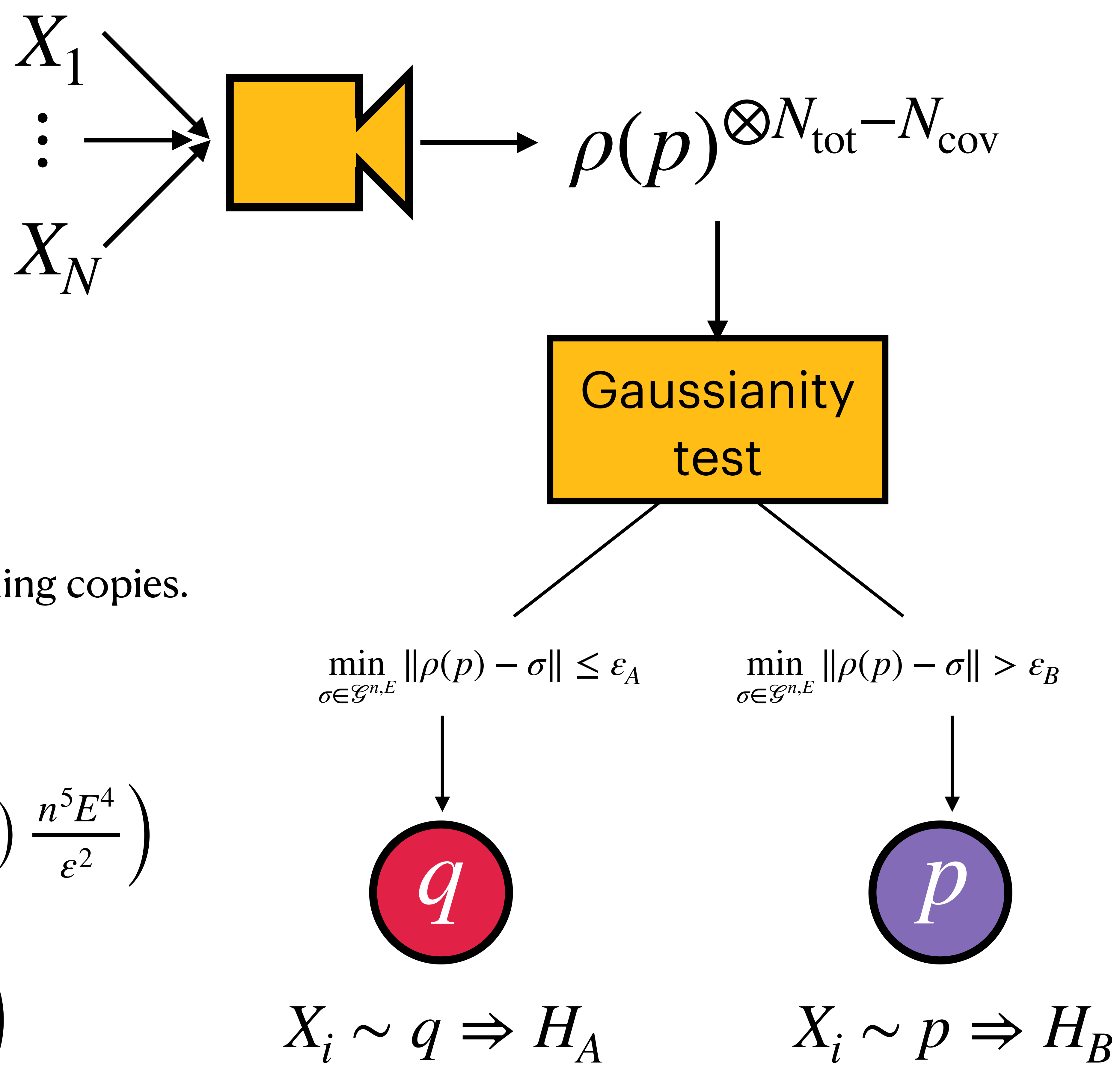


$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

Step 2: Gaussianity test





$$\mathrm{Tr}[\rho(q)\hat{E}^2] \le n^2 E^2$$

$$\mathrm{Tr}[\rho(p)\hat{E}^2] \le n^2 E^2$$

$$\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right) \frac{n^5 E^4}{\varepsilon^2}\right)$$

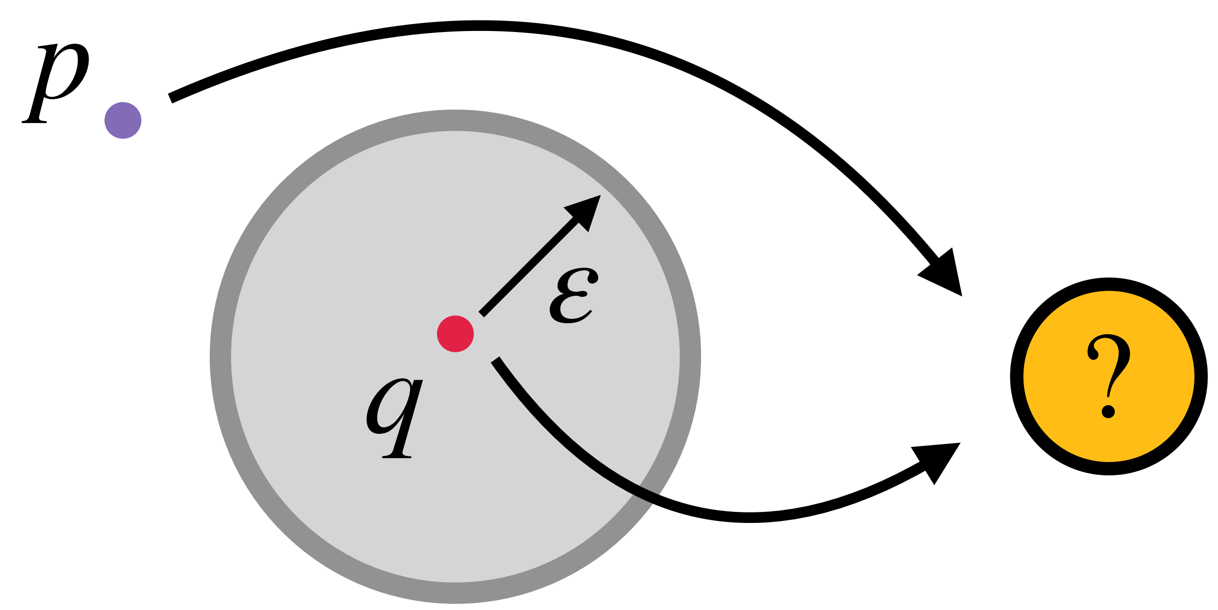
$$\nu = \Omega(E)$$

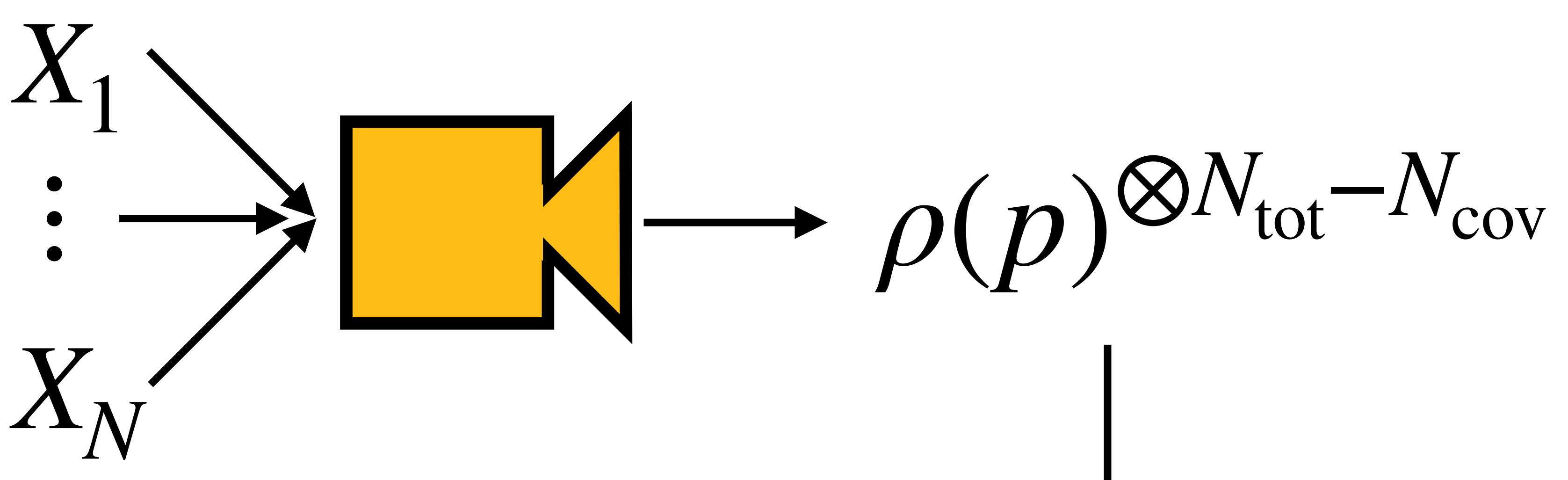
Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \le \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

$$\begin{aligned} N_{\mathrm{Gauss}} &= N_{\mathrm{tot}} - N_{\mathrm{cov}} \\ &= \Omega\left(\frac{1}{\varepsilon^2 \|q\|_2}\right) - O\left(\log\left(n^2\right) \frac{n^5 E^4}{\varepsilon^2}\right) \\ &= \Omega\left(\frac{(\nu+1)^n}{\varepsilon^2}\right) = \Omega\left(\frac{E^n}{\varepsilon^2}\right) \end{aligned}$$

Step 2: Gaussianity test





$$\text{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2$$

$$\text{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2$$

$$\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|\,k\rangle\langle k\,|$$

Otherwise, if $\|\tilde{V}(p)-V(q)\|_2 \leq \varepsilon_V$,

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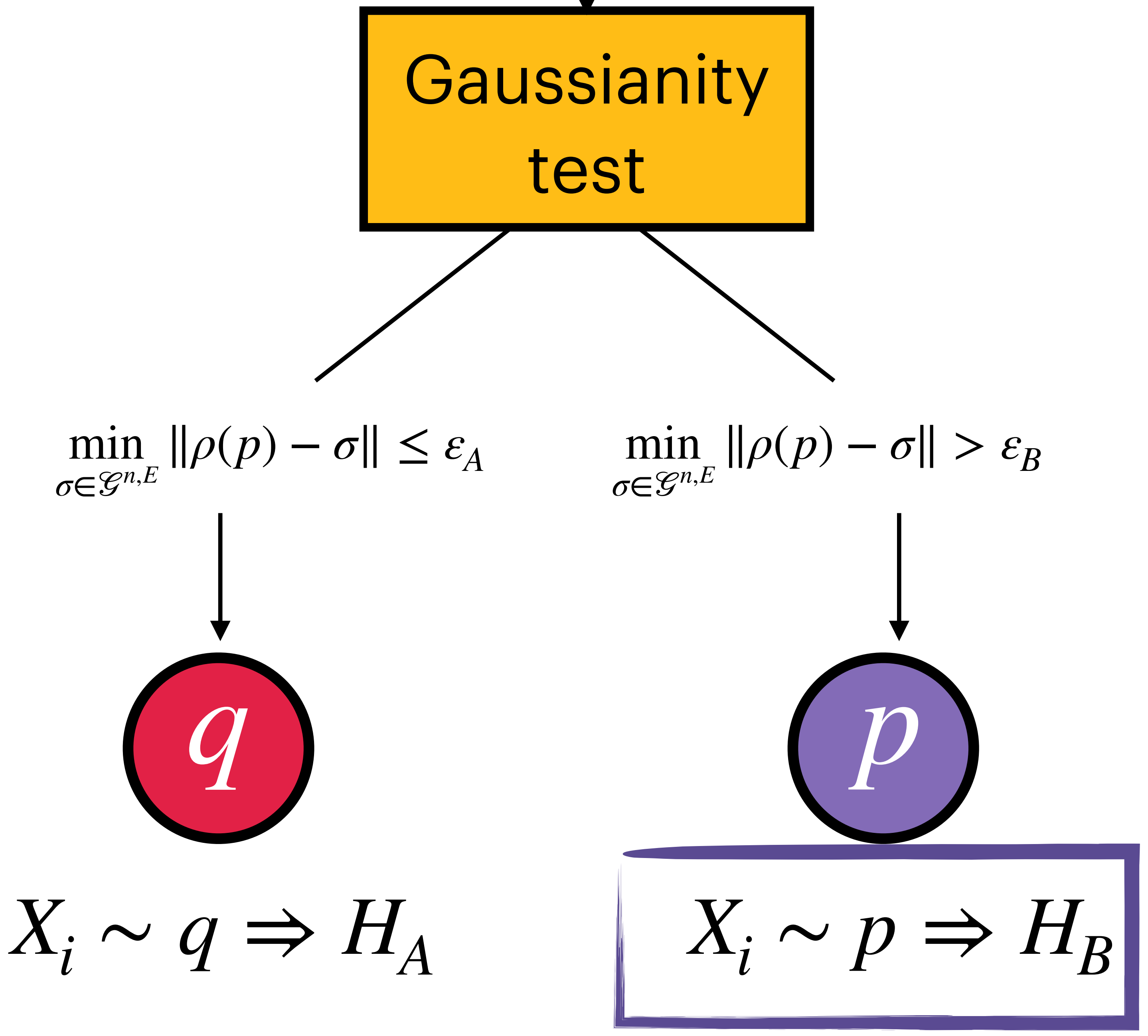
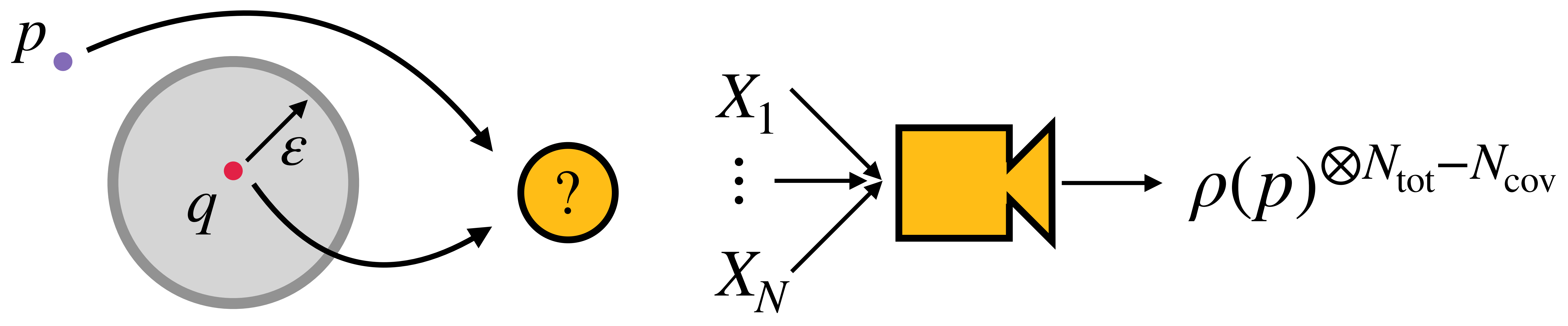


$$N_{\text{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

$$\begin{aligned} N_{\text{Gauss}} &= N_{\text{tot}} - N_{\text{cov}} \\ &= \Omega\left(\frac{1}{\varepsilon^2\|q\|_2}\right) - O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right) \\ &= \Omega\left(\frac{(\nu+1)^n}{\varepsilon^2}\right) = \Omega\left(\frac{E^n}{\varepsilon^2}\right) \end{aligned}$$

Step 2: Gaussianity test



$$\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2 E^2$$

$$\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2 E^2$$

$$\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

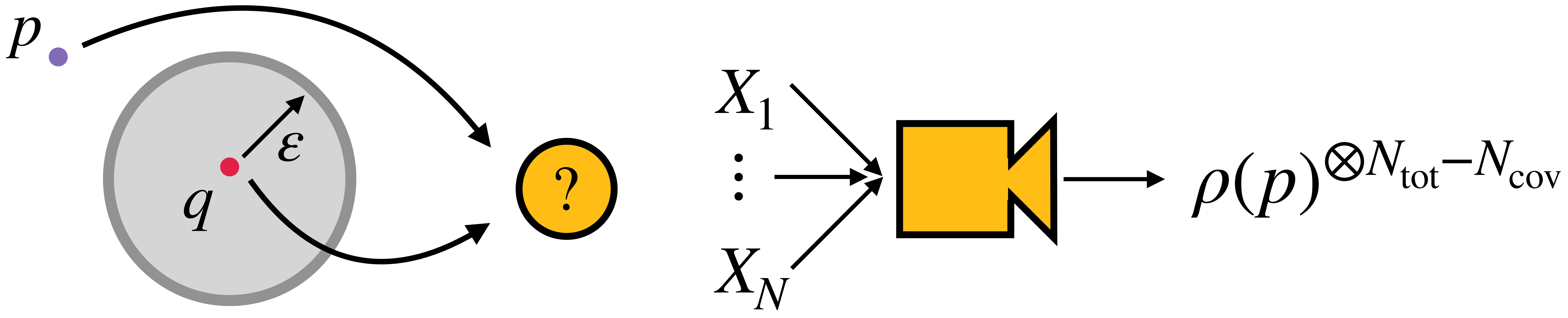
then we run the **Gaussianity test** on the remaining copies.

$$N_{\text{cov}} = O\left(\log\left(n^2\right) \frac{n^5 E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

$$\boxed{X_i \sim p \Rightarrow H_B}$$

Step 2: Gaussianity test



$$\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2$$
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Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

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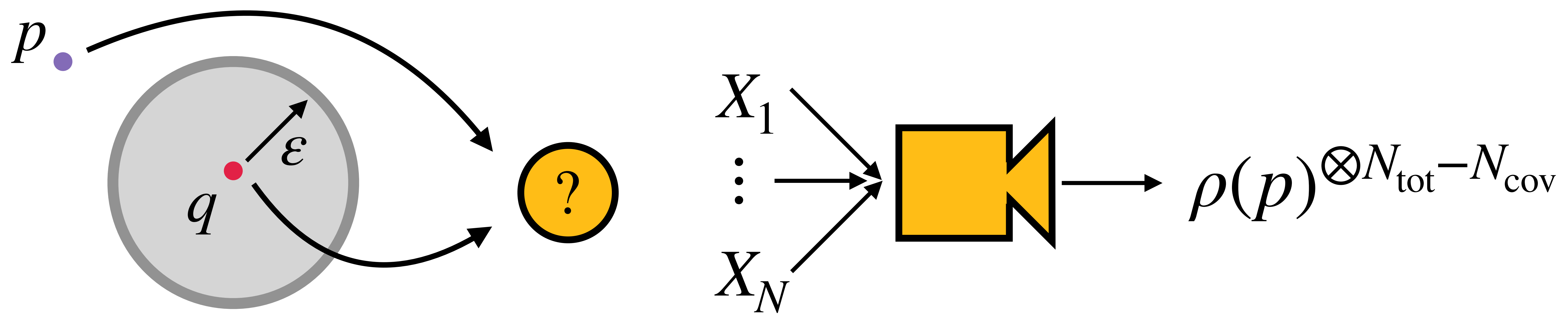
$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

$$\neg H_B \Rightarrow \frac{1}{2}\|p - q\|_1 < \varepsilon$$

$$X_i \sim p \Rightarrow H_B$$

Step 2: Gaussianity test



$$\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2 E^2$$

$$\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2 E^2$$

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$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5 E^4}{\varepsilon^2}\right)$$

$$\|p - q\|_1 = \|\rho(p) - \rho(q)\|_1 \leq \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1$$

$$\nu = \Omega(E)$$

$$\boxed{\neg H_B \Rightarrow \frac{1}{2}\|p - q\|_1 < \varepsilon}$$

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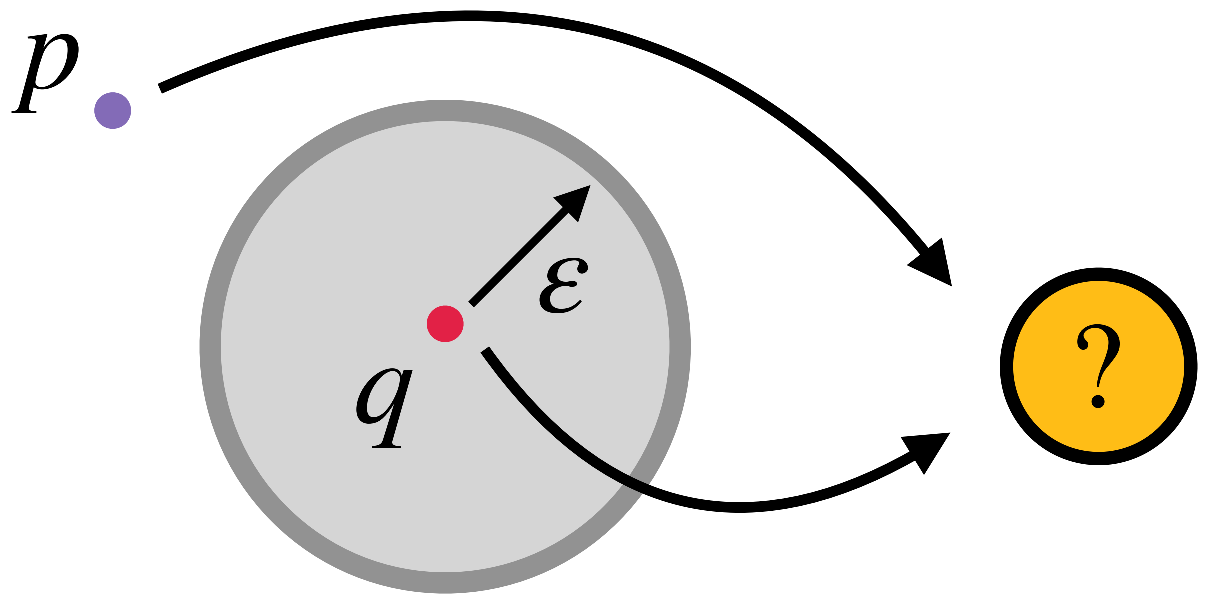
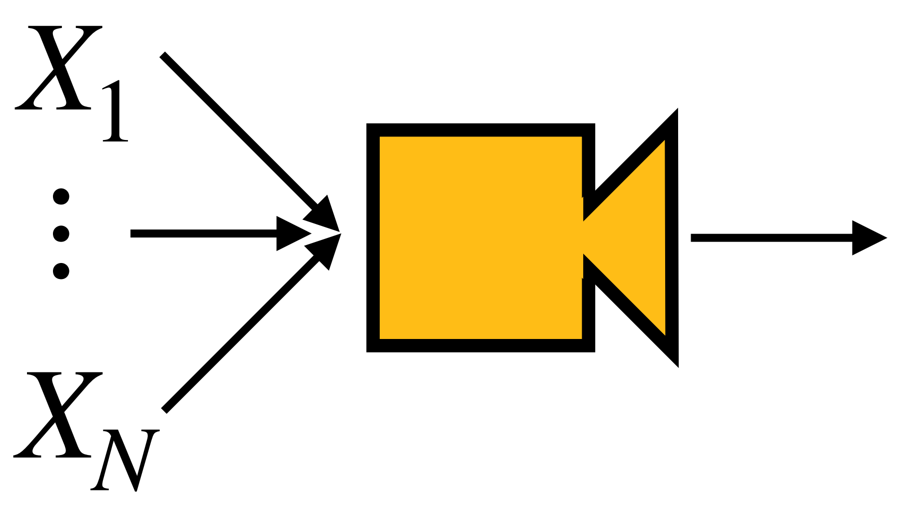
Step 2: Gaussianity test





$$\rho(p)^{\otimes N_{\mathrm{tot}} - N_{\mathrm{cov}}}$$

$$\boxed{\begin{array}{l} \mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2 \\ \mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2 \\ \frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon \end{array}}$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\begin{aligned} \|p-q\|_1 = \|\rho(p)-\rho(q)\|_1 &\leq \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1 \\ &\leq c_{nE}\left(\min_{\sigma\in\mathcal{G}_{E,\mathrm{mixed}}} \|\rho(p)-\sigma\|_1\right)^{1/2} + \|G(\rho(p))-\rho(q)\|_1 \end{aligned}$$

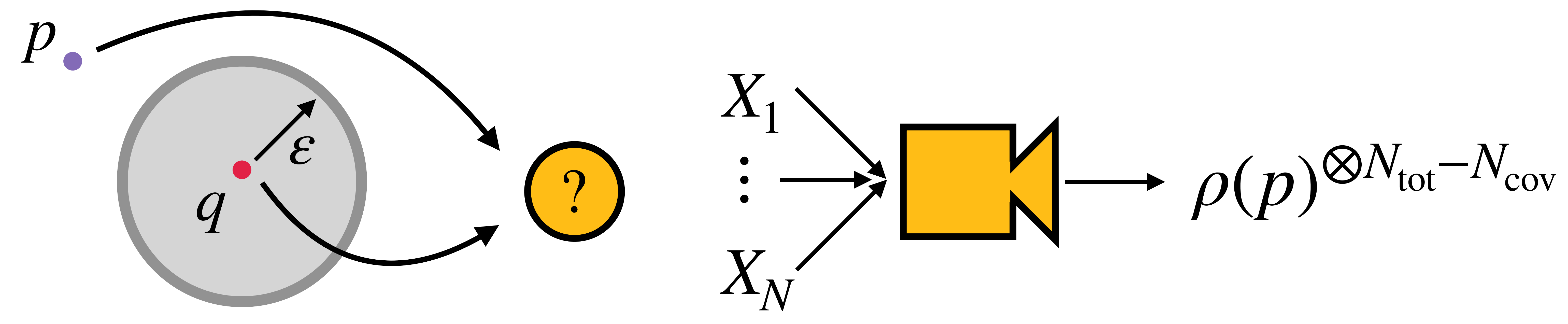
$$\nu = \Omega(E)$$

$$\boxed{\neg H_B \Rightarrow \frac{1}{2}\|p-q\|_1 < \varepsilon}$$

$$c_{nE} = \Theta\left((nE)^2\right)$$

$$\boxed{X_i \sim p \Rightarrow H_B}$$

Step 2: Gaussianity test



$$\boxed{\begin{aligned} &\mathrm{Tr}[\rho(q)\hat{E}^2] \le n^2E^2 \\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \le n^2E^2 \\ &\frac{1}{2}\|\rho(p)-\rho(q)\|_1 > \varepsilon \end{aligned}}$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \le \varepsilon_V$,

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$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\begin{aligned} \|p-q\|_1 = \|\rho(p)-\rho(q)\|_1 &\le \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1 \\ &\le c_{nE}\left(\min_{\sigma\in\mathcal{G}_{E,\mathrm{mixed}}}\|\rho(p)-\sigma\|_1\right)^{1/2} + \|G(\rho(p))-\rho(q)\|_1 \\ &\le c_{nE}\sqrt{2\varepsilon_B} + \frac{1+\sqrt{3}}{4}\max(\mathrm{Tr}V(p), \mathrm{Tr}V(q))\,\|V(p)-V(q)\|_\infty \end{aligned}$$

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$$\boxed{\neg H_B \Rightarrow \frac{1}{2}\|p-q\|_1 < \varepsilon}$$

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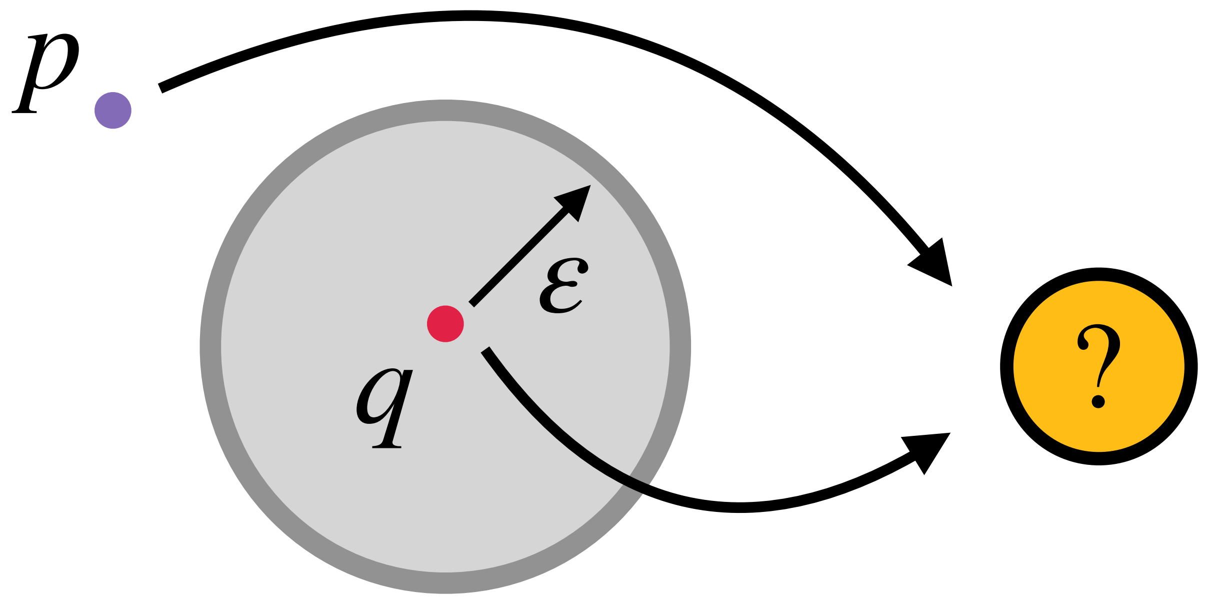
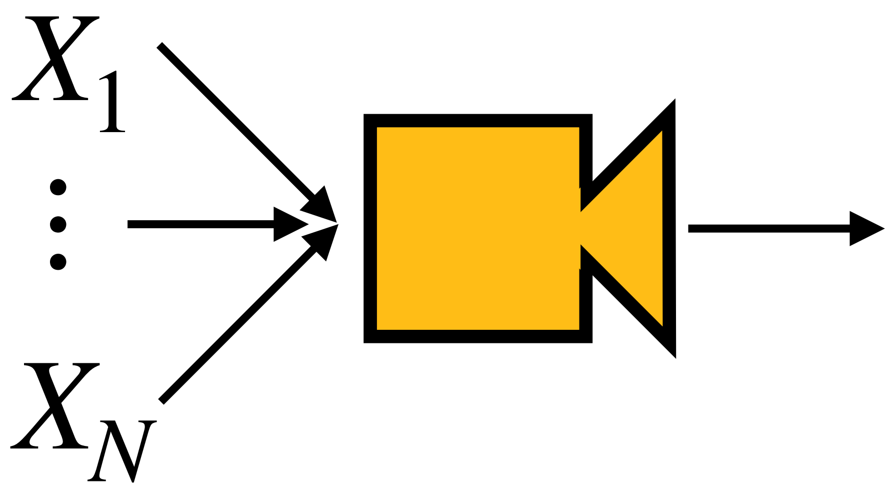
Step 2: Gaussianity test





$$\rho(p)^{\otimes N_{\text{tot}} - N_{\text{cov}}}$$

$$\boxed{\begin{aligned} &\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2 E^2 \\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2 E^2 \\ &\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon \end{aligned}}$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

$$N_{\text{cov}} = O\left(\log\left(n^2\right)\frac{n^5 E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

$$c_{nE} = \Theta\left((nE)^2\right)$$

Upper bound between Gaussian states.

$$\frac{1}{2}\|\rho(V, \mathbf{m}) - \rho(W, \mathbf{t})\|_1 \leq \frac{1+\sqrt{3}}{8}\max(\mathrm{Tr}V, \mathrm{Tr}W)\,\|V - W\|_\infty$$

he remaining copies.

$$\|p - q\|_1 = \|\rho(p) - \rho(q)\|_1 \leq \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1$$

$$\leq c_{nE}\left(\min_{\sigma \in \mathcal{G}_{E,\text{mixed}}} \|\rho(p) - \sigma\|_1\right)^{1/2} + \|G(\rho(p)) - \rho(q)\|_1$$

$$\leq c_{nE}\sqrt{2\varepsilon_B} + \frac{1+\sqrt{3}}{4}\max(\mathrm{Tr}V(p), \mathrm{Tr}V(q))\,\|V(p) - V(q)\|_\infty$$

$$\boxed{\neg H_B \Rightarrow \frac{1}{2}\|p - q\|_1 < \varepsilon}$$

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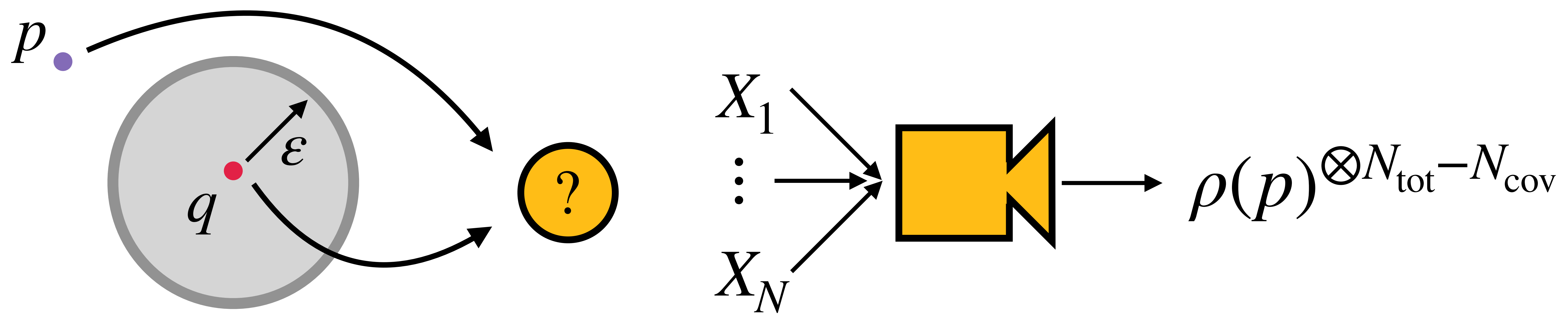
Step 2: Gaussianity test



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$$\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon$$

$$\rho(p) := \sum_{k\in\mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \le \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

$$\begin{aligned}\|p-q\|_1 = \|\rho(p)-\rho(q)\|_1 &\le \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1 \\ &\le c_{nE}\Big(\min_{\sigma\in\mathcal{G}_{E,\mathrm{mixed}}} \|\rho(p)-\sigma\|_1\Big)^{1/2} + \|G(\rho(p)) - \rho(q)\|_1 \\ &\le c_{nE}\sqrt{2\varepsilon_B} + \frac{1+\sqrt{3}}{4}\max(\mathrm{Tr}V(p), \mathrm{Tr}V(q))\,\|V(p)-V(q)\|_\infty\end{aligned}$$

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

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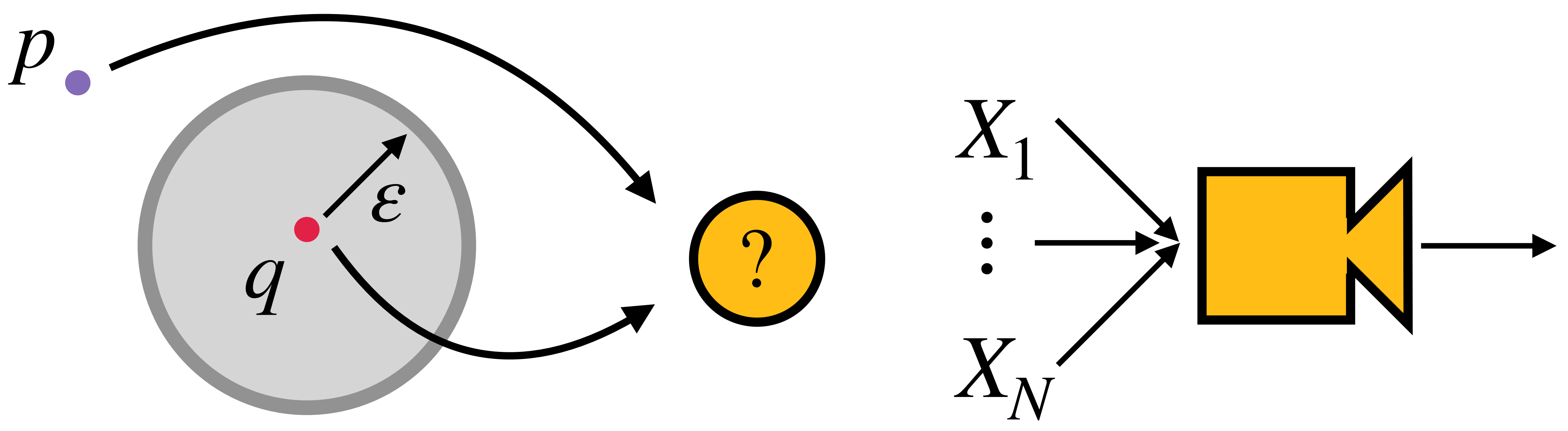
Step 2: Gaussianity test



$$\rho(p)^{\otimes N_{\mathrm{tot}} - N_{\mathrm{cov}}}$$

$$\boxed{\begin{aligned} &\mathrm{Tr}[\rho(q)\hat{E}^2] \leq n^2E^2 \\ &\mathrm{Tr}[\rho(p)\hat{E}^2] \leq n^2E^2 \\ &\frac{1}{2}\|\rho(p) - \rho(q)\|_1 > \varepsilon \end{aligned}}$$

$$\rho(p) := \sum_{k \in \mathbb{N}^n} p(k)\,|k\rangle\langle k|$$

$$N_{\mathrm{cov}} = O\left(\log\left(n^2\right)\frac{n^5E^4}{\varepsilon^2}\right)$$

$$\nu = \Omega(E)$$

$$c_{nE} = \Theta\left((nE)^2\right)$$

$$\varepsilon \equiv 8c_{nE}^2\varepsilon_B$$

Otherwise, if $\|\tilde{V}(p) - V(q)\|_2 \leq \varepsilon_V$,

then we run the **Gaussianity test** on the remaining copies.

$$\begin{aligned}
\|p - q\|_1 = \|\rho(p) - \rho(q)\|_1 &\leq \|\rho(p) - G(\rho(p))\|_1 + \|G(\rho(p)) - \rho(q)\|_1 \\
&\leq c_{nE}\left(\min_{\sigma \in \mathcal{G}_{E,\mathrm{mixed}}} \|\rho(p) - \sigma\|_1\right)^{1/2} + \|G(\rho(p)) - \rho(q)\|_1 \\
&\leq c_{nE}\sqrt{2\varepsilon_B} + \frac{1+\sqrt{3}}{4}\max(\mathrm{Tr}V(p), \mathrm{Tr}V(q))\,\|V(p) - V(q)\|_\infty \\
&\leq \frac{1}{2}\varepsilon + (1+\sqrt{3})nE\|V(p) - V(q)\|_2 \leq \frac{1}{2}\varepsilon + (1+\sqrt{3})nE(2\varepsilon_V) < \varepsilon
\end{aligned}$$

$$\boxed{\neg H_B \Rightarrow \frac{1}{2}\|p - q\|_1 < \varepsilon}$$

$$\boxed{X_i \sim p \Rightarrow H_B}$$

Outlook

Results.

- two protocols to test "pure Gaussianity" (symmetry, learning) with a polynomial number of copies of ρ with respect to the system size n and with local (or even single copy) measurements;
- hardness of testing "mixed Gaussianity" , even when having access to a quantum memory.

Open questions.

- optimal sample complexity?
- hardness even in the case ε_B constant.

$\left(\text{This is the case for relative entropy } \min_{\sigma \in \mathcal{G}_E^{\text{mixed}}} D(\rho \| \sigma).\right)$