Efficient learning of bosonic Gaussian unitaries

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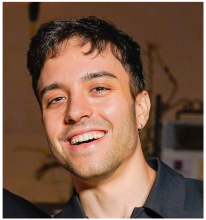

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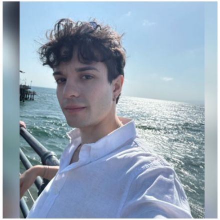

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Contents

Introduction

Motivation: Learning Theory in ∞-Dimensional Quantum Systems

Quantum Learning Theory

$\Downarrow$

Extensively developed
for **finite**-dimensional systems

Continuous-Variable Systems

$\Downarrow$

Bosonic systems, Quantum optics
Dimension $= \infty$

Quantum *state* tomography $=$ *Learning unknown quantum states*
Quantum *process* tomography $=$ *Learning unknown quantum channels*

Gap in the Literature

Despite many (heuristic) tomography methods in quantum optics, the literature lacks *"rigorous performance guarantees"*.

Motivation: Learning Theory in ∞-Dimensional Quantum Systems

- CV systems are indispensable in both experimental and theoretical quantum information science.
- **However, their ∞-dimensional structure introduces unique mathematical and statistical challenges.**
- Recent works have begun to explore learning in this regime:
  - Quantum state tomography
  - Trace-distance and energy-independent learning
  - Hamiltonian structure estimation
- Highlights both potential and limitations of existing techniques.

Motivation: Bosonic Systems

- Appear in:
  - Gaussian boson sampling (quantum advantage)
  - GKP codes (fault tolerance)
  - Quantum communication and cryptography
  - Quantum metrology and analog simulation
- Candidate platforms for building universal quantum computers
- **Precise learning algorithms are vital for calibration and benchmarking!**

Warm-up: Quantum State Tomography

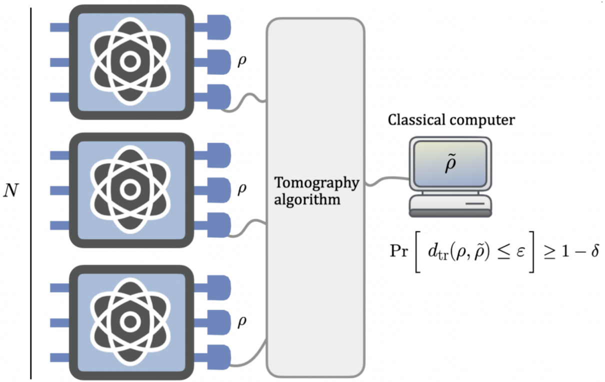


Image source: [Fig. 4, arXiv:2405.01431]

$\rho \in \mathcal{S} =$ some subset of the set of quantum states

Warm-up: Quantum State Tomography

Problem 1 (Quantum state tomography)

Given N copies of the (unknown) state $\rho \in \mathcal{S}$, the goal is to output $\tilde{\rho}$ such that

$$\Pr\left[d_{\mathrm{tr}}(\tilde{\rho}, \rho) \leq \epsilon\right] \geq 1 - \delta.$$

Definition

The **sample complexity** $N(\mathcal{S}, \epsilon, \delta)$ is the minimum N satisfying Problem 1.

Example 1

(i) $\mathcal{S} :=$ {qu$D$it states} $\Longrightarrow N(\mathcal{S}, \epsilon, \delta) = \tilde{\Theta}(D^2/\epsilon^2)$

(ii) $\mathcal{S} :=$ {qu$D$it pure states} $\Longrightarrow N(\mathcal{S}, \epsilon, \delta) = \tilde{\Theta}(D/\epsilon^2)$

n-qubit states $\Rightarrow$ $D = 2^n$ $\Rightarrow$ **Tomography is inefficient (without any assumptions)**

And in continuous-variable systems (where $D = \infty$), how would that work?

Warm-up: Quantum State Tomography

Without any additional prior assumptions, tomography is impossible in ∞-dimensional systems. In laboratory settings, bosonic systems have bounded energy.

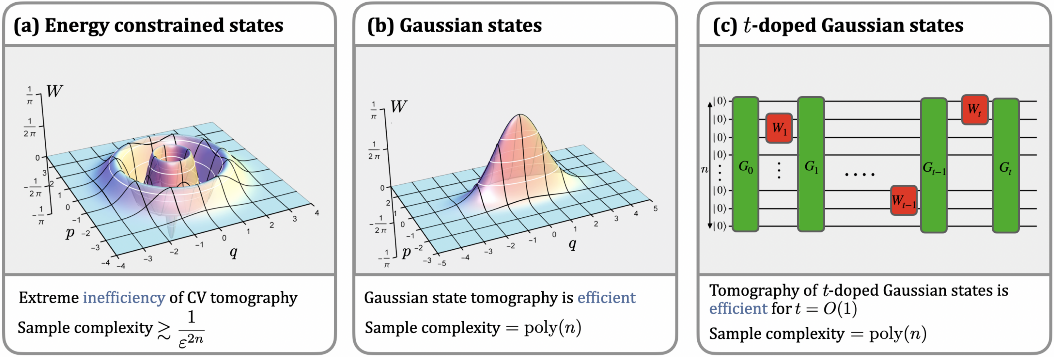
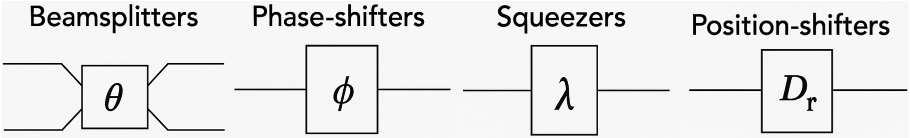


Image source: [Fig. 1, arXiv:2405.01431]

Warm-up: From State Learning to Process Learning

- Beyond learning quantum *states*, learning quantum *processes* is crucial.
- Quantum process tomography enables full characterization of transformations.
- **Learning arbitrary CV processes is extremely expensive.**
- Thus, attention must be restricted to structured subclasses.
  $\Rightarrow$ This work: Bosonic Gaussian Unitaries!



The Status of Learning "Structured" Quantum Objects

State Learning

	Basic	**Doped**
Stabilizer	[AG08], [Mon17]	[GIKL23], [LOH23], [HG23]
Fermionic Gaussian	[AG21], [O'G23], [B**M**EL25]	[**M**H24]
Bosonic Gaussian	[**MM**B+24]	[**MM**B+24]

References abbreviated as in the paper "Mildly-Interacting Fermionic Unitaries are Efficiently Learnable."

The Status of Learning "Structured" Quantum Objects

Unitary Learning

	Basic	**Doped**
Clifford	[Low09], [LOLH23]	OPEN PROBLEM
Fermionic Gaussian	[ODMZ22]	[**Vis**25], [AMG25]
Bosonic Gaussian	**This work!**	OPEN PROBLEM

References abbreviated as in the paper "Mildly-Interacting Fermionic Unitaries are Efficiently Learnable."

Our Central Question

Core Motivation

Can we design a computationally efficient algorithm for learning bosonic Gaussian unitaries to small worst-case error measured by a physically motivated distance?

- We provide an affirmative answer with rigorous theoretical guarantees!

Preliminaries

Notation

- $\mathbb{R}, \mathbb{N}, \mathbb{C}$: sets of real, natural, and complex numbers.
- $[n] \coloneqq \{1, \ldots, n\}$ for integer $n \geq 1$.
- $\mathbb{1}$: identity operator
- δ_{ij}: Kronecker delta ($= 1$ if $i = j$, else 0).

- For a matrix M:

$$\|M\|_\infty \text{ (operator norm)}, \quad \|M\|_2 \text{ (Hilbert–Schmidt norm)}, \quad \|M\|_1 \text{ (trace norm)}.$$

Gaussian Distribution

- For mean vector $\mathbf{m} \in \mathbb{R}^k$ and covariance $V \in \mathbb{R}^{k\times k}$:

$$\mathcal{N}(\mathbf{m}, V)\text{: Gaussian distribution on } \mathbb{R}^k.$$

- Probability density:

$$\mathcal{N}(\mathbf{m}, V)(\mathbf{x}) = \frac{e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^\top V^{-1}(\mathbf{x}-\mathbf{m})}}{(2\pi)^{k/2}\sqrt{\det V}}.$$

Symplectic and Orthogonal Groups

- Real symplectic group:

$$\mathrm{Sp}_{2m}(\mathbb{R}) = \{S \in \mathbb{R}^{2m\times 2m} : S^\top \Omega S = \Omega\}.$$

- Canonical symplectic form:

$$\Omega = \bigoplus_{i=1}^{m} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- Orthogonal group:

$$\mathrm{O}(n) = \{Q \in \mathbb{R}^{n\times n} : Q^\top Q = \mathbb{1}\}.$$

Continuous-Variable (CV) Systems

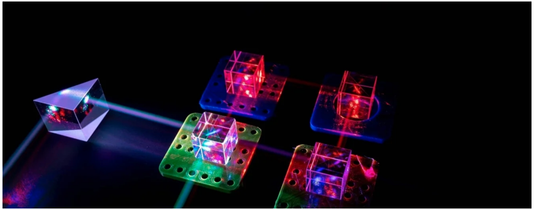


Image source: `https://www.azooptics.com/Article.aspx?ArticleID=1756`

- A continuous-variable (CV) system is a quantum system associated with the m-mode Hilbert space $L^2(\mathbb{R}^m)$, consisting of all square-integrable complex functions over $\mathbb{R}^m$.
- The number of modes $m \in \mathbb{N}$ plays the role of the *system size*, analogous to the number of qudits in discrete-variable (DV) systems.

Continuous-Variable (CV) Systems

- 1 *mode* $\iff$ 1 qudit with $d = \infty$
- Hilbert space: $\mathcal{H} = \mathrm{Span}\{ |0\rangle, |1\rangle, \ldots, |d\rangle, |d+1\rangle, \ldots\}$
  ($|0\rangle$: vacuum state, $|d\rangle$: d photons)
- A *CV system* consists of m modes (i.e., m qu$d$its with $d = \infty$).
- System size $\leftrightarrow$ number of modes m.

- m-mode quantum state $\leftrightarrow$ density operator on $L^2(\mathbb{R}^m)$.
- m-mode unitary $\leftrightarrow$ unitary operator acting on $L^2(\mathbb{R}^m)$.

Quadrature operators:

$$\hat{\mathbf{R}} := (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_m, \hat{p}_m)^\top.$$

Gaussian Unitaries

- $\hat{G}$ is a *Gaussian unitary* if $\hat{G} = e^{-i\hat{H}}$, for some (quadratic) Hamiltonian $\hat{H} := (\hat{\mathbf{R}} - \mathbf{m})^\top h(\hat{\mathbf{R}} - \mathbf{m})$, where $h \in \mathbb{R}^{2m\times 2m}$ is positive definite and $\mathbf{m} \in \mathbb{R}^{2m}$.
- The set of m-mode Gaussian unitaries is in one-to-one correspondence with pairs $(\mathbf{r}, S)$, where $\mathbf{r} \in \mathbb{R}^{2m}$ and $S \in \mathrm{Sp}_{2m}(\mathbb{R})$.
    - $\mathbf{r}$: *displacement vector*, S: *symplectic matrix*.
    - There is a one-to-one correspondence with symplectic matrices via the relation $U_S \leftrightarrow S$ defined by $U^\dagger\, \hat{\mathbf{R}}_j\, U = \sum_k S_{jk}\, \hat{\mathbf{R}}_k$.
- Any Gaussian unitary admits the decomposition

$$\hat{G} := G_{\mathbf{r},S} = D_{\mathbf{r}} U_S,$$

where $D_{\mathbf{r}} := e^{-i\mathbf{r}^\top \Omega \hat{\mathbf{R}}}$ is the displacement operator, and U_S is the symplectic Gaussian unitary associated with S. Their action on quadratures is

$$D_{\mathbf{r}}^\dagger \hat{\mathbf{R}} D_{\mathbf{r}} = \hat{\mathbf{R}} + \mathbf{r}\mathbb{1}, \qquad U_S \hat{\mathbf{R}} U_S^\dagger = S\hat{\mathbf{R}}.$$

Gaussian States

- A state ρ is a *Gaussian state* if

$$\rho = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}[e^{-\beta \hat{H}}]}$$

for some $\beta \in (0, \infty]$ and some (quadratic) Hamiltonian $\hat{H}$ of the form $\hat{H} := (\hat{\mathbf{R}} - \mathbf{m})^\top h (\hat{\mathbf{R}} - \mathbf{m})$, where $h \in \mathbb{R}^{2m \times 2m}$ is positive definite and $\mathbf{m} \in \mathbb{R}^{2m}$.

- Any pure Gaussian state $|\psi\rangle$ is of the form $|\psi\rangle = \hat{G} |0\rangle$, for some Gaussian unitary $\hat{G}$.

Characterization: uniquely determined by

- First moment $\mathbf{m}(\rho) = \mathrm{Tr}[\hat{\mathbf{R}}\rho] \in \mathbb{R}^{2m}$,
- Covariance matrix $V(\rho) = \mathrm{Tr}\left[\left\{\hat{\mathbf{R}} - \mathbf{m}(\rho)\mathbb{1},\ (\hat{\mathbf{R}} - \mathbf{m}(\rho)\mathbb{1})^\top\right\} \rho\right] \in \mathbb{R}^{2m \times 2m}$.

Transformations:

$$\mathbf{m}(D_{\mathbf{r}} \rho D_{\mathbf{r}}^\dagger) = \mathbf{m}(\rho) + \mathbf{r}, \qquad V(D_{\mathbf{r}} \rho D_{\mathbf{r}}^\dagger) = V(\rho),$$
$$\mathbf{m}(U_S \rho U_S^\dagger) = S\mathbf{m}(\rho), \qquad V(U_S \rho U_S^\dagger) = S V(\rho) S^\top.$$

Examples of Gaussian States

- **Coherent states** $|\mathbf{m}\rangle$:

$$\mathbf{m}(|\mathbf{m}\rangle) = \mathbf{m}, \qquad V(|\mathbf{m}\rangle) = \mathbb{1}.$$

  The case $\mathbf{m} = 0$ is the *vacuum state.*

- **Single-mode squeezed state:**

$$V(|z_{\text{in}}\rangle) = \begin{pmatrix} z_{\text{in}} & 0 \\ 0 & z_{\text{in}}^{-1} \end{pmatrix}.$$

- **Two-mode squeezed vacuum (TMSV):**

$$V(|\nu\rangle) = \begin{pmatrix} (2\nu - 1)\mathbb{1} & 2\sqrt{\nu(\nu-1)}\,\sigma_z \\ 2\sqrt{\nu(\nu-1)}\,\sigma_z & (2\nu-1)\mathbb{1} \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Measurements: Homodyne & Heterodyne Sampling

Setup:

- n-mode Gaussian state ρ with first moment $\mathbf{m} \in \mathbb{R}^{2n}$ and covariance $V \in \mathbb{R}^{2n \times 2n}$.
- Quadrature operator vector: $\hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^\top$.

Homodyne measurement:

- Measures a subset of quadratures (either all $\hat{x}_j$ or all $\hat{p}_j$).
- Outcomes follow a multivariate Gaussian law:

$$\mathbf{x} \sim \mathcal{N}(\mathbf{m}_x, V_{xx}/2), \qquad \mathbf{p} \sim \mathcal{N}(\mathbf{m}_p, V_{pp}/2),$$

where

$$\mathbf{m}_x = (m_{2j-1})_{j\in[n]}, \quad \mathbf{m}_p = (m_{2j})_{j\in[n]}.$$

Measurements: Homodyne & Heterodyne Sampling

Heterodyne measurement:

- Provides simultaneous (but noisy) information about both the position and momentum quadratures of each bosonic mode.
- For a Gaussian state ρ with first-moment vector $\mathbf{m} \in \mathbb{R}^{2m}$ and covariance matrix $V \in \mathbb{R}^{2m \times 2m}$, the heterodyne outcomes are distributed according to a classical Gaussian law,

$$\text{Heterodyne}(\rho) \sim \mathcal{N}\left(\mathbf{m}, \frac{V + \mathbb{1}}{2}\right),$$

Distance Measures in Quantum Learning

- In quantum learning, error quantification requires distances with clear physical meaning.
- For states, the standard metric is the **trace distance**:

$$\frac{1}{2}\left\|\rho_1 - \rho_2\right\|_1 .$$

- Operational meaning (Holevo–Helstrom theorem):

$$P_{\mathrm{succ}} = \frac{1}{2}\left(1 + \frac{1}{2}\left\|\rho_1 - \rho_2\right\|_1\right).$$

$\Rightarrow$ If two states are close in trace distance, their expectation values for all bounded observables are close.

Diamond Distance for Quantum Channels

- For channels Φ_1, Φ_2:

$$\frac{1}{2}\left\|\Phi_1 - \Phi_2\right\|_\diamond = \sup_{\rho_{AA'}} \frac{1}{2}\left\|\mathrm{Id}_A \otimes \Phi_1(\rho_{AA'}) - \mathrm{Id}_A \otimes \Phi_2(\rho_{AA'})\right\|_1 .$$

- Operational meaning (Holevo–Helstrom theorem):

$$P_{\mathrm{succ}} = \frac{1}{2}\left(1 + \frac{1}{2}\left\|\Phi_1 - \Phi_2\right\|_\diamond\right).$$

- **Problem in continuous-variable (CV) systems:**
  - Optimization involves input states of unbounded energy.
  - Hence, $\|\cdot\|_\diamond$ becomes unphysical.
- Example: For any distinct beam splitters $\mathcal{U}_{\lambda_1}, \mathcal{U}_{\lambda_2}$,

$$\frac{1}{2}\left\|\mathcal{U}_{\lambda_1} - \mathcal{U}_{\lambda_2}\right\|_\diamond = 1,$$

even when $\lambda_1 \approx \lambda_2$.

Energy-Constrained Diamond Distance

- Introduced by Winter and Shirokov.
- For parameter $\bar{n} > 0$ (mean photon number),

$$\frac{1}{2}\left\|\Phi_1 - \Phi_2\right\|_{\diamond,\bar{n}} := \sup_{\substack{\rho_{AA'}: \\ \mathrm{Tr}[\rho_{AA'}(\hat{N}_A \otimes \mathbb{1}_{A'})] \leq \bar{n}}} \frac{1}{2}\left\|\mathrm{Id}_A \otimes \Phi_1(\rho_{AA'}) - \mathrm{Id}_A \otimes \Phi_2(\rho_{AA'})\right\|_1 .$$

- **Operational meaning:**

$$P_{\mathrm{succ}} = \frac{1}{2}\left(1 + \frac{1}{2}\left\|\Phi_1 - \Phi_2\right\|_{\diamond,\bar{n}}\right),$$

  under the constraint $\mathrm{Tr}[\rho\hat{N}] \leq \bar{n}$.
- Physically meaningful for CV channels:
  - Restricts optimization to realizable states with bounded mean energy.
  - Used as the default error metric in CV channel tomography.

Gaussian unitary tomography problem

Tomography of Gaussian Unitaries

Goal: Estimate an unknown Gaussian unitary $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ using access to channel queries, with performance measured in the *energy-constrained diamond distance* $\|\cdot\|_{\diamond,\bar{n}}$.

Gaussian structure:

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S, \quad \mathbf{r} \in \mathbb{R}^{2m}, \quad S \in \mathrm{Sp}_{2m}(\mathbb{R}).$$

Problem parameters:

- $\bar{n}$: photon-number constraint in the **error metric**.
- $\bar{n}_{\mathrm{in}}$: photon-number constraint on **input states**.
- z: **squeezing bound**, i.e., $\|S\|_\infty \leq z$.

Formal Problem Statement

Problem: Tomography of Gaussian unitaries

Let $m \in \mathbb{N}$, $N_{\mathrm{tot}} \in \mathbb{N}$, $z \geq 1$, $\bar{n} > 0$, $\bar{n}_{\mathrm{in}} > 0$, $\varepsilon \in (0,1)$, and $\delta \in (0,1)$ be known parameters. Design a quantum algorithm that

- **Given:** black-box access to an unknown m-mode Gaussian unitary $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$, where $S \in \mathrm{Sp}_{2m}(\mathbb{R})$ satisfies $\|S\|_\infty \leq z$;
- **Using:** at most N_{tot} queries to $G_{\mathbf{r},S}$ and only input states with mean photon number at most $\bar{n}_{\mathrm{in}}$;
- **Outputs:** estimators $\tilde{\mathbf{r}} \in \mathbb{R}^{2m}$ and $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$; the corresponding Gaussian unitary channel $\tilde{\mathcal{G}} := \mathcal{D}_{\tilde{\mathbf{r}}} \circ \mathcal{U}_{\tilde{S}}$ approximates the true channel $\mathcal{G} = \mathcal{D}_{\mathbf{r}} \circ \mathcal{U}_S$ and satisfies

$$\Pr\left[\frac{1}{2}\left\|\tilde{\mathcal{G}} - \mathcal{G}\right\|_{\diamond,\bar{n}} \leq \varepsilon\right] \geq 1 - \delta,$$

  where the diamond norm is taken with respect to the mean photon number constraint $\bar{n}$.

Algorithm Overview

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

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$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

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$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Algorithm Overview

$$\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon_r$$

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Algorithm Overview

$$\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon_r$$

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\frac{1}{2}\|\tilde{\mathcal{G}} - \mathcal{G}\|_{\diamond,\bar{n}} \leq \varepsilon$$

$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Algorithm Overview

We specify a Gaussian unitary

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S, \quad \mathbf{r} \in \mathbb{R}^{2m},\ S \in \mathrm{Sp}_{2m}(\mathbb{R}).$$

Four-stage reconstruction pipeline:

1. **Symplectic estimation:** Use coherent probes and heterodyne detection to estimate S with operator-norm error $\leq \varepsilon_S$.
2. **Symplectic regularization:** Apply square-root correction to enforce exact symplecticity, obtaining $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$ with
$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(z^2 \varepsilon_S), \quad \|S\|_\infty \leq z.$$
3. **Displacement estimation:** Use squeezed Gaussian probes and $U_{\tilde{S}^{-1}}$ precompensation to estimate $\tilde{\mathbf{r}}$ with Euclidean error ε_r.
4. **End-to-end guarantee:** The reconstructed channel $\tilde{\mathcal{G}} = \mathcal{D}_{\tilde{\mathbf{r}}} \circ \mathcal{U}_{\tilde{S}}$ satisfies
$$\Pr\left[\frac{1}{2}\|\tilde{\mathcal{G}} - \mathcal{G}\|_{\diamond,\bar{n}} \leq \varepsilon\right] \geq 1 - \delta.$$

Learning the symplectic component

Learning the Symplectic Component S

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

Learning the Symplectic Component S

Objective: Estimate the symplectic matrix S in the Gaussian unitary

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S.$$

Heterodyne detection:

$$\text{Heterodyne}(G_{\mathbf{r},S} \left|\mathbf{m}\right\rangle) \sim \mathcal{N}\left(\mathbf{r} + S\mathbf{m},\ \frac{SS^\top + \mathbb{1}}{2}\right).$$

- For a coherent input $\left|\mathbf{m}\right\rangle$, mean shifts as $\mathbf{r} + S\mathbf{m}$.
- Covariance encodes quadratic structure: $SS^\top$.

Vacuum input:

$$Y_0 \sim \mathcal{N}\left(\mathbf{r},\ \frac{SS^\top + \mathbb{1}}{2}\right).$$

Columnwise Probing and Estimation of S

Columnwise probing:

- For each $i \in [2m]$, prepare $|\eta e_i\rangle$, where e_i is the i-th standard basis vector.
- Output distribution:

$$Y_i \sim \mathcal{N}\left(\mathbf{r} + \eta S_i,\ \frac{SS^\top + \mathbb{1}}{2}\right),$$

where S_i is the i-th column of S.

Estimation:

$$\frac{Y_i - Y_0}{\eta} \text{ is an unbiased estimator of } S_i.$$

- Stack all column estimates $\Rightarrow$ preliminary matrix $\hat{S}$.
- $\hat{S}$ **may not be perfectly symplectic.**
- Apply projection to obtain valid $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$.

Symplectic Learning with Vacuum-Shared Inputs

$$|0\rangle \xrightarrow{\text{Heterodyne}} \bar{Y}_0 := \frac{1}{N_S}\sum_{k=1}^{N_S} Y_0^{(k)}$$

$$|\eta e_i\rangle \xrightarrow{\text{Heterodyne}} \bar{Y}_i := \frac{1}{N_S}\sum_{k=1}^{N_S} Y_i^{(k)}$$

$$\underbrace{\begin{pmatrix} \square \\ \square \\ \vdots \\ \square \\ \square \end{pmatrix}}_{\hat{S}_i} := \frac{\bar{Y}_i - \bar{Y}_0}{\eta}$$

$$S = \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

$$\Pr\left[\|\hat{S} - S\|_\infty \leq \varepsilon\right] \geq 1 - \delta$$

$$\hat{S} = \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \notin \mathrm{Sp}_{2m}(\mathbb{R})$$

$$\begin{array}{ccccc} \hat{S}_1 & \hat{S}_2 & & \hat{S}_{2m-1} & \hat{S}_{2m} \end{array}$$

Symplectic Learning with Vacuum-Shared Inputs

Query complexity for learning the symplectic part with vacuum-shared inputs

Let $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ be a Gaussian unitary on m bosonic modes with symplectic matrix $S \in \mathrm{Sp}_{2m}(\mathbb{R})$. Fix the heterodyne measurement model for coherent input probes $|\mathbf{m}\rangle$, for which the outcome is a random vector

$$Y \sim \mathcal{N}\left(\mathbf{r} + S\mathbf{m},\, \Sigma\right), \qquad \Sigma := \frac{SS^\top + \mathbb{1}}{2}.$$

In particular, let Y_0 denote the outcome distribution for the vacuum probe $|0\rangle$, and Y_i the outcome distribution for the input coherent state $|\eta e_i\rangle$ with $\eta > 0$ and $i \in [2m]$. From N_S independent heterodyne samples of each probe, we form the empirical means

$$\bar{Y}_0 := \frac{1}{N_S} \sum_{k=1}^{N_S} Y_0^{(k)}, \qquad \bar{Y}_i := \frac{1}{N_S} \sum_{k=1}^{N_S} Y_i^{(k)}.$$

Symplectic Learning with Vacuum-Shared Inputs

Query complexity for learning the symplectic part with vacuum-shared inputs

We then define the estimators

$$\hat{S}_i := \frac{\bar{Y}_i - \bar{Y}_0}{\eta}, \qquad \hat{S} := [\hat{S}_1, \ldots, \hat{S}_{2m}].$$

Then, for every $\varepsilon > 0$ and $\delta \in (0,1)$, if

$$N_S \geq \frac{4m\|S\|_\infty^2 \left(\sqrt{2m} + \sqrt{2\log(2m/\delta)}\right)^2}{\eta^2 \varepsilon^2},$$

we have

$$\Pr\left[\|\hat{S} - S\|_\infty \leq \varepsilon\right] \geq 1 - \delta.$$

In particular, the total number of queries to the unitary $G_{\mathbf{r},S}$ is $(2m+1)N_S$.

Symplectic Learning with Vacuum-Shared Inputs (Proof)

Let $n := 2m$. Since $SS^\top \succcurlyeq 0$ and $I \succ 0$, we have $\Sigma \succ 0$, and hence $\Sigma^{\pm 1/2}$ are well defined.

Step 1 (Columnwise distribution). For each $i \in [n]$, we obtain

$$\bar{Y}_0 \sim \mathcal{N}\left(\mathbf{r},\ \frac{\Sigma}{N_S}\right),$$
$$\bar{Y}_i \sim \mathcal{N}\left(\mathbf{r} + \eta S_i,\ \frac{\Sigma}{N_S}\right),$$

with the two batches independent. Consequently,

$$\hat{S}_i - S_i = \frac{1}{\eta} \cdot \left[\bar{Y}_i - (\mathbf{r} + \eta S_i) - (\bar{Y}_0 - \mathbf{r})\right] \sim \mathcal{N}\left(0,\ \frac{2\Sigma}{\eta^2 N_S}\right).$$

Preliminaries: Concentration Inequalities

Lemma (Estimation of first moments)

Let $\{\hat{x}_i\}_{i=1}^N$ *be* N *i.i.d. samples from an* n*-dimensional Gaussian distribution* $\mathcal{N}(\mu, \Sigma)$*. Define the empirical mean as* $\hat{\mu} \coloneqq \frac{1}{N}\sum_{i=1}^N \hat{x}_i$*. Then, for every* $\delta \in (0,1)$*,*

$$\Pr\left[\|\hat{\mu} - \mu\|_2 \le \frac{\chi_{n,\delta}}{\sqrt{N}}\sqrt{\|\Sigma\|_\infty}\right] \ge 1 - \delta,$$

where $\chi_{n,\delta} \coloneqq \sqrt{n} + \sqrt{2\log(1/\delta)}$*.*

Symplectic Learning with Vacuum-Shared Inputs (Proof)

Step 2 (Columnwise ℓ_2 control). N_S i.i.d. samples defining $\bar{Y}_i$ and $\bar{Y}_0$, we obtain that for any $\delta_i \in (0,1)$,

$$\Pr\left[\|\hat{S}_i - S_i\|_2 \leq \sqrt{\frac{2\|\Sigma\|_\infty}{\eta^2 N_S}} \cdot \left(\sqrt{n} + \sqrt{2\log(1/\delta_i)}\right)\right] \geq 1 - \delta_i.$$

Choosing $\delta_i = \delta/n$ and applying the union bound over $i \in [n]$ yields, for all $i \in [n]$,

$$\Pr\left[\|\hat{S}_i - S_i\|_2 \leq \sqrt{\frac{2\|\Sigma\|_\infty}{\eta^2 N_S}} \cdot \left(\sqrt{n} + \sqrt{2\log(n/\delta)}\right),\ \forall i \in [n]\right] \geq 1 - \delta.$$

Symplectic Learning with Vacuum-Shared Inputs (Proof)

Step 3 (From columns to operator norm). Let $E := \hat{S} - S = [e_1, \dots, e_n]$ with $e_i = \hat{S}_i - S_i$. Then

$$\begin{aligned} \|\hat{S} - S\|_\infty \leq\ & \|E\|_2 \\ =\ & \left(\sum_{i=1}^{n} \|e_i\|_2^2 \right)^{1/2} \\ \leq\ & \sqrt{n} \cdot \sqrt{\frac{2\|\Sigma\|_\infty}{\eta^2 N_S}} \cdot \left(\sqrt{n} + \sqrt{2\log(n/\delta)}\right) \end{aligned}$$

Combining this, we obtain

$$\Pr\left[\|\hat{S} - S\|_\infty \leq \sqrt{\frac{2\|\Sigma\|_\infty}{\eta^2 N_S}} \cdot \sqrt{n} \cdot \left(\sqrt{n} + \sqrt{2\log(n/\delta)}\right) \right] \geq 1 - \delta.$$

Symplectic Learning with Vacuum-Shared Inputs (Proof)

Step 4 (Solve for N_S and upper bound $\|\Sigma\|_\infty$). To ensure that $\|\hat{S} - S\|_\infty \leq \varepsilon$, it suffices that

$$N_S \geq \frac{2\|\Sigma\|_\infty}{\eta^2 \varepsilon^2} \cdot n\left(\sqrt{n} + \sqrt{2\log(n/\delta)}\right)^2.$$

Since S is symplectic, we have $\|S\|_\infty \geq 1$ and

$$\|\Sigma\|_\infty \leq \frac{1}{2}\left(\|S\|_\infty^2 + 1\right) \leq \|S\|_\infty^2,$$

Substituting $n = 2m$ completes the proof. By construction, the total number of queries is $(2m+1)N_S$.

$$\boxed{N_S \geq \frac{4m\|S\|_\infty^2\left(\sqrt{2m} + \sqrt{2\log(2m/\delta)}\right)^2}{\eta^2 \varepsilon^2}}$$

Symplectic Learning with Symmetric Probes

Idea: We present an alternative algorithm for learning the symplectic component S.

Key modification: Use *symmetric probes* $|\pm\eta e_i\rangle$ instead of the vacuum-shared setup.

Motivation:

- In the vacuum-shared case, columnwise samples are **dependent**.
  - $\Rightarrow$ Operator-norm error bounded via **union bound** over columns.
- Using symmetric probes makes all columnwise samples **independent**.
  - $\Rightarrow$ Enables direct use of the **Gaussian operator-norm tail bound**.

Symplectic Learning with Symmetric Probes

$$|+\eta e_i\rangle \xrightarrow{\text{Heterodyne}} \bar{Y}_i^{(+)} := \frac{1}{N_S}\sum_{k=1}^{N_S} Y_{i,+}^{(k)}$$

$$|-\eta e_i\rangle \xrightarrow{\text{Heterodyne}} \bar{Y}_i^{(-)} := \frac{1}{N_S}\sum_{k=1}^{N_S} Y_{i,-}^{(k)}$$

$$\underbrace{\begin{pmatrix} \square \\ \square \\ \vdots \\ \square \\ \square \end{pmatrix}}_{\hat{S}_i} := \frac{\bar{Y}_i^{(+)} - \bar{Y}_i^{(-)}}{2\eta}$$

$$S = \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

$$\Pr\left[\|\hat{S} - S\|_\infty \leq \varepsilon\right] \geq 1 - \delta$$

$$\hat{S} = \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \notin \mathrm{Sp}_{2m}(\mathbb{R})$$

$$\begin{array}{ccccc} \hat{S}_1 & \hat{S}_2 & & \hat{S}_{2m-1} & \hat{S}_{2m} \end{array}$$

Symplectic Learning with Symmetric Probes

Lemma (Tail bound for Gaussian operator norm)

Let $A \in \mathbb{R}^{m\times n}$ have i.i.d. $\mathcal{N}(0,1)$ entries. Then, for all $t \geq 0$,

$$\Pr\left[\|A\|_\infty \geq \sqrt{m} + \sqrt{n} + t\right] \leq e^{-t^2/2}.$$

Implication for symmetric-probe learning (Gaussian operator-norm tail bound):

$$N_S \geq \frac{\|S\|_\infty^2 \left(2\sqrt{2m} + \sqrt{2\log(1/\delta)}\right)^2}{2\eta^2\varepsilon^2}, \qquad \text{total queries: } 4mN_S.$$

Comparison (vacuum-shared case, columnwise ℓ_2 control):

$$N_S \geq \frac{4m\|S\|_\infty^2 \left(\sqrt{2m} + \sqrt{2\log(2m/\delta)}\right)^2}{\eta^2\varepsilon^2}, \qquad \text{total queries: } (2m+1)N_S.$$

Regularization of Learned Symplectic Matrices

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Symplectic Rounding: Motivation

Problem: The estimators $\hat{S}$ obtained from previous procedures are not guaranteed to be symplectic.

Goal: Design an efficient rounding algorithm to find a nearby $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$ such that

$$\left\|\tilde{S} - S\right\|_\infty \leq 9z^2\varepsilon, \quad \text{given } \left\|\hat{S} - S\right\|_\infty \leq \varepsilon,\ \|S\|_\infty \leq z.$$

Idea: Use a symplectic analogue of the polar decomposition.

$$T := -\Omega\hat{S}^\top\Omega\hat{S}, \qquad Q := \sqrt{T}, \qquad \tilde{S} := Q^{-1}\hat{S}.$$

If Q exists and is well-defined, $\tilde{S}$ will be symplectic.

Regularization of Learned Symplectic Matrices

$$\hat{S}$$

$$\notin \mathrm{Sp}_{2m}(\mathbb{R})$$

Regularization of Learned Symplectic Matrices

$$\left(\sqrt{-\Omega\,\hat{S}^{\top}\,\Omega\,\hat{S}}\right)^{-1}\cdot\,\hat{S}$$

$$\in \mathrm{Sp}_{2m}(\mathbb{R})$$

Regularization of Learned Symplectic Matrices

$$\left(\sqrt{-\begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix} \begin{array}{c} \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \\ \begin{array}{ccccc} \hat{s}_1 & \hat{s}_2 & & \hat{s}_{2m-1} & \hat{s}_{2m} \end{array} \end{array}^{\top} \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix} \begin{array}{c} \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \\ \begin{array}{ccccc} \hat{s}_1 & \hat{s}_2 & & \hat{s}_{2m-1} & \hat{s}_{2m} \end{array} \end{array}}\right)^{-1} \cdot \begin{array}{c} \begin{pmatrix} \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \square & \square & \cdots & \square & \square \\ \square & \square & \cdots & \square & \square \end{pmatrix} \\ \begin{array}{ccccc} \hat{s}_1 & \hat{s}_2 & & \hat{s}_{2m-1} & \hat{s}_{2m} \end{array} \end{array}$$

$$\in \mathrm{Sp}_{2m}(\mathbb{R})$$

Symplecticity of the Rounded Matrix

Our novel algorithmic tool: $\tilde{S} = \left(\sqrt{-\Omega \hat{S}^\top \Omega \hat{S}}\right)^{-1} \hat{S}$ is *symplectic!*

Proposition. If $Q = \sqrt{T}$ exists and is well-defined, then $\tilde{S} = Q^{-1}\hat{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$.

Proof sketch:

$$T = -\Omega \hat{S}^\top \Omega \hat{S}, \qquad Q^2 = T.$$

- Show that $Q^\top = \Omega Q \Omega^{-1}$ (principal root invariance).
- Then compute:

$$\tilde{S}^\top \Omega \tilde{S} = (Q^{-1})^\top \hat{S}^\top \Omega \hat{S} Q^{-1} = \Omega Q^{-1} Q^2 Q^{-1} = \Omega.$$

Thus, $\tilde{S}$ is symplectic.

Properties of the Principal Square Root

Well-definedness. If $\|A - I\|_\infty < 1$, then $\sqrt{A}$ exists and is unique. All eigenvalues of A have positive real part.

Lipschitz continuity near the identity. For A with $\|A - I\|_\infty < 1/2$,

$$\|\sqrt{A} - I\|_\infty \leq (2 - \sqrt{2})\|A - I\|_\infty.$$

Proof idea:

- Use the integral representation of $\sqrt{I + B}$.
- Apply submultiplicativity and Neumann-series bounds.

These results ensure $Q = \sqrt{T}$ is both well-defined and stable for small perturbations.

Preliminaries: Linear Algebra and Matrix Analysis

Existence and uniqueness of the principal square root

Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$. Then there exists a unique square root X of A whose eigenvalues all lie in the open right half-plane. This X is called the principal square root of A, and we write $X = \sqrt{A} = A^{1/2}$. If A is real, then $\sqrt{A}$ is also real.

Neumann series

Let A be a bounded linear operator on a normed space with $\|A\|_\infty < 1$. Then the series

$$\sum_{k=0}^{\infty} A^k$$

converges in operator norm, to $(\mathbb{1} - A)^{-1}$. In particular, $(\mathbb{1} - A)$ is invertible with inverse given by the Neumann series above.

Preliminaries: Linear Algebra and Matrix Analysis

Matrix p-th root representation and perturbation

Let $A \in \mathbb{R}^{n\times n}$ have no eigenvalues on the closed negative real axis and $p > 1$. Then for any $r \geq (2\,\|A\|_\infty)^{1/p}$,

$$A^{1/p} = \frac{p\sin(\pi/p)}{\pi} \cdot A\left(\int_0^r (t^p\mathbb{1} + A)^{-1}dt + \int_r^\infty (t^p\mathbb{1} + A)^{-1}dt\right),$$

where

$$\left\|\int_r^\infty (t^p\mathbb{1} + A)^{-1}dt\right\|_\infty \leq \frac{2r^{1-p}}{p-1}.$$

Moreover, if $P \in \mathbb{R}^{n\times n}$ is such that $A + P$ has no eigenvalues on $\mathbb{R}^-$, then

$$(A+P)^{1/p} - A^{1/p} = \frac{p\sin(\pi/p)}{\pi}\int_0^\infty t^p\,(t^p\mathbb{1} + A + P)^{-1}\,P\,(t^p\mathbb{1} + A)^{-1}\,dt.$$

Error Bound for Symplectic Rounding

Lemma (Error bound for symplectic rounding)

Suppose $\left\|\hat{S} - S\right\|_\infty \leq \varepsilon$, $\|S\|_\infty \leq z$, *and* $(2z+1)\varepsilon < 1/2$. *Then* $\left\|\tilde{S} - S\right\|_\infty \leq 9z^2\varepsilon$.

Proof sketch:

- Show $\|T - I\|_\infty \leq (2z+1)\varepsilon < 1/2$.
- Apply Lipschitz bound: $\|Q - I\|_\infty \leq (2-\sqrt{2})(2z+1)\varepsilon$.
- Use Neumann expansion for Q^{-1}:

$$\|Q^{-1}\|_\infty \leq 1 + 2(2-\sqrt{2})(2z+1)\varepsilon.$$

- Combine via triangle inequality and submultiplicativity:

$$\|\tilde{S} - S\|_\infty \leq 9z^2\varepsilon.$$

Conclusion: $\tilde{S}$ is symplectic and remains close to S with controlled operator-norm error.

Learning a Regularized Symplectic Matrix: Comparison

Estimate symplectic matrix $S \in \mathrm{Sp}_{2m}(\mathbb{R})$ in $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$, then apply symplectic regularization to obtain $\tilde{S}$.

$$\Pr\left[\|\tilde{S} - S\|_\infty \leq \tau\right] \geq 1 - \delta.$$

Method	Heterodyne shots per probe N_S	Total queries to $G_{\mathbf{r},S}$
Vacuum-shared inputs	$\dfrac{324\, m z^6 \left(\sqrt{2m} + \sqrt{2\log(2m/\delta)}\right)^2}{\eta^2 \tau^2}$	$(2m+1)N_S$
Symmetric probes	$\dfrac{81\, z^6 \left(2\sqrt{2m} + \sqrt{2\log(1/\delta)}\right)^2}{2\eta^2 \tau^2}$	$4mN_S$

Learning the displacement component

Learning the Displacement Component

$$\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon_r$$

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\|\hat{S} - S\|_\infty \leq \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \leq \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Displacement Learning with TMSV States

Goal: Estimate $\mathbf{r}$ in $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ after learning the symplectic part $\tilde{S}$.

Protocol:

1. Prepare the $2m$-mode product TMSV state:

$$|\nu\rangle^{\otimes m} = U_{S_\nu} |0\rangle^{\otimes 2m}, \qquad S_\nu = \begin{pmatrix} \sqrt{\nu} I & \sqrt{\nu - 1} Z \\ \sqrt{\nu - 1} Z & \sqrt{\nu} I \end{pmatrix}, \quad Z = \bigoplus_{i=1}^{m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Apply in sequence:

$$U_{\tilde{S}^{-1}} \rightarrow G_{\mathbf{r},S} \rightarrow U_{S_\nu}^\dagger.$$

Intuition:

- Entanglement with the auxiliary system amplifies $\mathbf{r}$ by $\sqrt{\nu}$.
- For large ν, heterodyne detection yields high signal-to-noise ratio.

Moments of the Output State

Let $\Delta = \tilde{S}^{-1}S - I$. After the full protocol, the output has moments:

$$\mathbf{m}(U_{S_\nu}^\dagger G_{\mathbf{r},S} U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m}) = (\sqrt{\nu}\,\mathbf{r},\ -\sqrt{\nu - 1} Z\mathbf{r}),$$
$$V(U_{S_\nu}^\dagger G_{\mathbf{r},S} U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m}) = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix},$$

where

$$A = I + \nu(\Delta + \Delta^\top) + \nu(2\nu + 1)\Delta\Delta^\top,$$
$$B = I - (\nu - 1)(\Delta + \Delta^\top) + (\nu + 1)(2\nu + 1)\Delta\Delta^\top,$$
$$C = \left[-(2\nu - 1)\sqrt{\nu(\nu - 1)}\,\Delta\Delta^\top - \sqrt{\nu(\nu - 1)}(\Delta^\top - \Delta)\right] Z.$$

Observation: If $\|\Delta\|_\infty = o(1/\nu)$, heterodyne detection on the first subsystem gives $\mathbf{r}$ with $o(1/\sqrt{\nu})$ error (high probability).

Query Complexity of Displacement Learning

Setting:

- Use heterodyne detection on the first $2m$ output modes.
- Each outcome:

$$Y^{(1)} \sim \mathcal{N}\left(\sqrt{\nu}\,\mathbf{r}, \Sigma^{(1)}\right), \quad \Sigma^{(1)} = \frac{A+I}{2}.$$

Estimator:

$$\tilde{\mathbf{r}} = \frac{\hat{\mu}^{(1)}}{\sqrt{\nu}}, \qquad \hat{\mu}^{(1)} = \frac{1}{N_r}\sum_{k=1}^{N_r} Y_k^{(1)}.$$

Guarantee: For every $\varepsilon > 0$ and $\delta \in (0,1)$, if

$$\boxed{N_r \geq \frac{\left(1 + \nu\|\Delta\|_\infty + \frac{3}{2}(\nu\|\Delta\|_\infty)^2\right)\left(\sqrt{2m} + \sqrt{2\log(1/\delta)}\right)^2}{\nu\varepsilon^2},}$$

then $\Pr[\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon] \geq 1 - \delta$.

$\Rightarrow$ High squeezing ($\nu \gg 1$) and accurate $\tilde{S}$ reduce the sample complexity as $O(1/\nu)$.

Learning the Displacement without TMSV States

Motivation:

- Two-mode squeezed vacuum states require entanglement between system and ancilla.
- We propose an alternative algorithm based only on **single-mode squeezing** and **homodyne detection**.

Setup:

- Given an estimated $\tilde{S}$ of the symplectic part of $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$.
- Define the deviation $\Delta = S\tilde{S}^{-1} - I$.
- Prepare $U_{\tilde{S}^{-1}} \left| z_{\text{in}} \right\rangle^{\otimes m}$, where

$$V(|z_{\text{in}}\rangle) = \bigoplus_{i=1}^{m} \begin{pmatrix} z_{\text{in}} & 0 \\ 0 & z_{\text{in}}^{-1} \end{pmatrix}.$$

Procedure:

$$U_{\tilde{S}^{-1}} \;\rightarrow\; G_{\mathbf{r},S} \;\rightarrow\; \text{homodyne detection (momentum or position).}$$

Learning Momentum and Position Components

(1) Momentum component:

- Input: $|z_{\text{in}}\rangle^{\otimes m}$ (momentum-squeezed state).
- Homodyne on momentum quadratures $\Rightarrow$

$$Y_p \sim \mathcal{N}\left(\mathbf{r}_p,\ \frac{V_{pp}}{2}\right), \quad V_{pp} = [(\Delta + I)V(|z_{\text{in}}\rangle)(\Delta + I)^\top]_{pp}.$$

- Covariance bound:

$$\boxed{\left\|\frac{V_{pp}}{2}\right\|_\infty \le \frac{1}{2}\left(\frac{(1+\|\Delta\|_\infty)^2}{z_{\text{in}}} + z_{\text{in}}\|\Delta\|_\infty^2\right)}.$$

(2) Position component:

- Input: $|z_{\text{in}}^{-1}\rangle^{\otimes m}$ (position-squeezed state).
- Homodyne on position quadratures $\Rightarrow$

$$Y_x \sim \mathcal{N}\left(\mathbf{r}_x,\ \frac{V_{xx}}{2}\right), \quad \text{with the same covariance bound as above.}$$

Query Complexity of Displacement Learning

Lemma. Let $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ act on m modes, and define $\Delta = \tilde{S}^{-1} S - I$. Fix squeezing $z_{\text{in}} \geq 1$. If

$$\boxed{N_r \geq \frac{2\left(\sqrt{2m} + \sqrt{2\log(2/\delta)}\right)^2}{\varepsilon^2} \left(\frac{(1 + \|\Delta\|_\infty)^2}{z_{\text{in}}} + z_{\text{in}} \|\Delta\|_\infty^2 \right),}$$

then using $2N_r$ queries to $G_{\mathbf{r},S}$ yields

$$\Pr[\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon] \geq 1 - \delta.$$

Comparison of Displacement Learning Methods

Goal: Estimate the displacement vector $\mathbf{r}$ in $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$.

$$\Pr[\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon] \geq 1 - \delta.$$

Method	Required samples N_r per probe	Total queries to $G_{\mathbf{r},S}$
TMSV probes	$N_r \geq \dfrac{\left(1 + \nu\|\Delta\|_\infty + \frac{3}{2}(\nu\|\Delta\|_\infty)^2\right)\left(\sqrt{2m} + \sqrt{2\log(1/\delta)}\right)^2}{\nu\,\varepsilon^2}$	N_r
single-mode squeezing	$N_r \geq \dfrac{2\left(\sqrt{2m} + \sqrt{2\log(2/\delta)}\right)^2}{\varepsilon^2}\left(\dfrac{(1 + \|\Delta\|_\infty)^2}{z_{\text{in}}} + z_{\text{in}}\|\Delta\|_\infty^2\right)$	$2N_r$

Experiment-Friendly Implementation (No Active Squeezing)

Goal: Realize the block

$$U_{S_\nu}^\dagger\, G_{\mathbf{r},S}\, U_{\tilde{S}^{-1}}\, |\nu\rangle^{\otimes m}$$

without any online active squeezing.

Naïve view: appears to require

(i) online application of the *active* Gaussian $U_{S_\nu}^\dagger$ before detection;

(ii) applying $U_{\tilde{S}^{-1}}$ directly to a squeezed input.

Resolution: both (i) and (ii) can be implemented exactly with passive linear optics, offline squeezing, and homodyne detection only.

Passive Realization of the Measurement Stage

Passive realization of heterodyne after an active Gaussian

For any Gaussian state ρ and Gaussian unitary U_S, the heterodyne statistics of $U_S \rho U_S^\dagger$ can be reproduced by the following *passive scheme:*

1. Prepare a squeezed-vacuum ancilla σ with covariance $S^{-1}S^{-\top}$.
2. Interfere ρ and σ on a balanced beam splitter.
3. Homodyne $\hat{x}$ on one arm and $\hat{p}$ on the other.
4. Record $\mathbf{q} = \sqrt{2}(\mathbf{x}, \mathbf{p})$ and output $S\mathbf{q}$.

Result: $S\mathbf{q}$ has the same distribution as the heterodyne outcome of $U_S \rho U_S^\dagger$.

Therefore, $U_{S_\nu}^\dagger$ never needs to be applied online.

Offline Implementation of $U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m}$

Rewriting the input preparation:

$$U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m} = U_{\tilde{S}^{-1} S_\nu} |0\rangle^{\otimes m} .$$

By Euler (Bloch–Messiah) decomposition:

$$\tilde{S}^{-1} S_\nu = O_1 Z O_2, \quad O_1, O_2 \in \mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{O}(2n),\ Z = \mathrm{diag}(z_1, z_1^{-1}, \ldots).$$

Hence

$$U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m} = U_{O_1} |Z\rangle ,$$

where $|Z\rangle$ is an offline-prepared squeezed state.

Implementation:

- Offline squeezing $|Z\rangle$ (input ancilla)
- Passive interferometer U_{O_1}
- Standard homodyne detection

$\rightarrow$ **Entire block realized with only passive optics and offline squeezing.**

End-to-end learning of Gaussian unitaries

End-to-End Learning of Gaussian Unitaries

$$\|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \le \varepsilon_r$$

$$G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$$

$$\frac{1}{2}\|\tilde{\mathscr{G}} - \mathscr{G}\|_{\diamond,\bar{n}} \le \varepsilon$$

$$\|\hat{S} - S\|_\infty \le \varepsilon_S$$

$$\|\tilde{S} - S\|_\infty \le \mathcal{O}(\varepsilon_S),\ \tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$$

Upper Bound on the Energy-Constrained Diamond Distance Between Gaussian Unitaries

The energy-constrained diamond norm admits quantitative continuity bounds for Gaussian unitary channels!

Bounds for displacement channels

Let $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^{2m}$, and let $\mathcal{D}_{\mathbf{r}_1}, \mathcal{D}_{\mathbf{r}_2}$ denote the displacement channels defined by $\mathcal{D}_{\mathbf{r}_1}(\rho) := D_{\mathbf{r}_1}\rho D_{\mathbf{r}_1}^{\dagger}$ and $\mathcal{D}_{\mathbf{r}_2}(\rho) := D_{\mathbf{r}_2}\rho D_{\mathbf{r}_2}^{\dagger}$, for all states ρ. Then, for every mean photon-number $\bar{n} \geq 0$,

$$\frac{1}{2}\left\|\mathcal{D}_{\mathbf{r}_1} - \mathcal{D}_{\mathbf{r}_2}\right\|_{\diamond,\bar{n}} \leq \sin\left(\min\left\{\frac{\left(\sqrt{\bar{n}} + \sqrt{\bar{n}+1}\right)}{\sqrt{2}} \cdot \left\|\mathbf{r}_1 - \mathbf{r}_2\right\|_2 ,\ \frac{\pi}{2}\right\}\right).$$

Upper Bound on the Energy-Constrained Diamond Distance Between Gaussian Unitaries

Bounds for symplectic Gaussian unitaries

Let $S_1, S_2 \in \mathrm{Sp}_{2m}(\mathbb{R})$, and let $\mathcal{U}_{S_1}, \mathcal{U}_{S_2}$ denote the Gaussian unitary channels defined by $\mathcal{U}_{S_1}(\rho) := U_{S_1} \rho U_{S_1}^{\dagger}$ and $\mathcal{U}_{S_2}(\rho) := U_{S_2} \rho U_{S_2}^{\dagger}$, for all states ρ. Then, for every mean photon-number $\bar{n} \geq 0$,

$$\frac{1}{2} \left\| \mathcal{U}_{S_1} - \mathcal{U}_{S_2} \right\|_{\diamond, \bar{n}} \leq \sqrt{(\sqrt{6} + \sqrt{10} + 5\sqrt{2m})(\bar{n}+1)}\, g\left(\left\| S_2^{-1} S_1 \right\|_{\infty} \right) \sqrt{\left\| S_2^{-1} S_1 - \mathbb{1} \right\|_2},$$

where $g(x) := \sqrt{\pi/(x+1)} + \sqrt{2x}$.

From Symplectic and Displacement Errors to Diamond Distance Error

Let $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ be a Gaussian unitary on m bosonic modes with symplectic matrix $S \in \mathrm{Sp}_{2m}(\mathbb{R})$ satisfying $\|S\|_\infty \leq z$. Let $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$ and $\tilde{\mathbf{r}} \in \mathbb{R}^{2m}$, and define

$$\varepsilon_S := \|\tilde{S} - S\|_\infty, \qquad \varepsilon_r := \|\tilde{\mathbf{r}} - \mathbf{r}\|_2.$$

Then, the energy-constrained diamond distance between $\tilde{\mathcal{G}} := \mathcal{D}_{\tilde{\mathbf{r}}} \circ \mathcal{U}_{\tilde{S}}$ and $\mathcal{G} := \mathcal{D}_{\mathbf{r}} \circ \mathcal{U}_S$ satisfies

$$\frac{1}{2}\|\tilde{\mathcal{G}} - \mathcal{G}\|_{\diamond,\bar{n}} \leq 12\sqrt{9\sqrt{2m}(\bar{n}+1)}\sqrt{z\sqrt{2m}\,\varepsilon_S} + \sqrt{2}\sqrt{z^2\bar{n}+1}\,\varepsilon_r,$$

where the diamond norm is taken with respect to the mean photon number constraint $\bar{n}$. In particular, for any $\varepsilon > 0$, if it holds that

$$\varepsilon_S \leq \frac{\varepsilon^2}{2592mz(\bar{n}+1)}, \qquad \varepsilon_r \leq \frac{\varepsilon}{2\sqrt{2}\sqrt{z^2\bar{n}+1}},$$

then $\frac{1}{2}\|\tilde{\mathcal{G}} - \mathcal{G}\|_{\diamond,\bar{n}} \leq \varepsilon$.

Summary of the Gaussian Unitary Learning Algorithm

Two-stage learning framework:

- *Stage 1 — Learn the symplectic component S:* $\varepsilon_S \leq \frac{\varepsilon^2}{2592\, m\, z\, (\bar{n}+1)}$
- *Stage 2 — Learn the displacement vector $\mathbf{r}$:* $\varepsilon_r \leq \frac{\varepsilon}{2\sqrt{2}\sqrt{z^2\bar{n}+1}}$

Available protocol options:

	Symplectic learning	**Displacement learning**
Option A	Vacuum-shared inputs	Two-mode squeezed vacuum
Option B	Symmetric probes	Two-mode squeezed vacuum
Option C	Vacuum-shared inputs	Single-mode squeezed states
Option D	Symmetric probes	Single-mode squeezed states

Chosen configuration:

- Vacuum-shared input protocol + Two-mode squeezed vacuum protocol
- Yields the best asymptotic query complexity for large input energy $\bar{n}_{\text{in}}$.

Learning Valid Symplectic and Displacement

Let $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$ be a Gaussian unitary on m bosonic modes with displacement $\mathbf{r} \in \mathbb{R}^{2m}$ and symplectic matrix $S \in \mathrm{Sp}_{2m}(\mathbb{R})$ satisfying $\|S\|_\infty \leq z$. Fix accuracy parameters $\varepsilon_S \in (0, 1/(2z))$, $\varepsilon_r \in (0,1)$ and failure probability $\delta \in (0,1)$. There exists a protocol that outputs an estimate $(\tilde{\mathbf{r}}, \tilde{S})$ such that

$$\Pr\left[\|\tilde{S} - S\|_\infty \leq \varepsilon_S \ \text{ and } \ \|\tilde{\mathbf{r}} - \mathbf{r}\|_2 \leq \varepsilon_r\right] \geq 1 - \delta,$$

where $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$. The total query complexity is $(2m+1)N_S + N_r$, where

$$N_S \geq \frac{324 m z^6 \left(\sqrt{2m} + \sqrt{2\log(2m/\delta)}\right)^2}{\eta^2 \varepsilon_S^2},$$

$$N_r \geq \frac{\left(1 + 2\nu z \varepsilon_S + 6(\nu z \varepsilon_S)^2\right)\left(\sqrt{2m} + \sqrt{\log(2/\delta)}\right)^2}{\nu \varepsilon_r^2},$$

with probe amplitude $\eta > 0$ and squeezing parameter $\nu \geq 1$.

Main Theorem: Learning Gaussian Unitaries in $\|\cdot\|_{\diamond,\bar{n}}$

Setup: Let $m\in\mathbb{N}$, $z\geq 1$, $\bar{n},\bar{n}_{\text{in}}>0$, $\varepsilon,\delta\in(0,1)$ and parameters $\eta,\varepsilon_S,\varepsilon_r>0$, $\nu\geq 1$ satisfying

$$\boxed{\eta\leq\sqrt{\bar{n}_{\text{in}}},\quad \nu\leq 1+\frac{\bar{n}_{\text{in}}}{2m},\quad \varepsilon_S\leq\frac{\varepsilon^2}{2592\,mz(\bar{n}+1)},\quad \varepsilon_r\leq\frac{\varepsilon}{2\sqrt{2}\sqrt{z^2\bar{n}+1}}.}$$

Algorithm parameters:

$$\boxed{\begin{aligned} N_S &\geq \frac{324\,m\,z^6\left(\sqrt{2m}+\sqrt{2\log(2m/\delta)}\right)^2}{\eta^2\varepsilon_S^2},\\ N_r &\geq \frac{\left(1+2\nu z\varepsilon_S+6(\nu z\varepsilon_S)^2\right)\left(\sqrt{2m}+\sqrt{\log(2/\delta)}\right)^2}{\nu\varepsilon_r^2}. \end{aligned}}$$

Main Theorem: Learning Gaussian Unitaries in $\|\cdot\|_{\diamond,\bar{n}}$

Algorithm guarantees:

- *Access:* black-box queries to an unknown m-mode Gaussian unitary $G_{\mathbf{r},S} = D_{\mathbf{r}} U_S$, with $\|S\|_\infty \leq z$.
- *Resources:*

$$N_{\mathrm{tot}} = (2m+1)N_S + N_r$$

  queries using only states with mean photon number $\leq \bar{n}_{\mathrm{in}}$.
- *Output:* estimators $\tilde{S} \in \mathrm{Sp}_{2m}(\mathbb{R})$, $\tilde{\mathbf{r}} \in \mathbb{R}^{2m}$ such that

$$\Pr\left[\frac{1}{2}\|\tilde{\mathcal{G}} - \mathcal{G}\|_{\diamond,\bar{n}} \leq \varepsilon\right] \geq 1 - \delta, \quad \tilde{\mathcal{G}} = \mathcal{D}_{\tilde{\mathbf{r}}} \circ \mathcal{U}_{\tilde{S}}.$$

Asymptotic Query Complexity of the Algorithm

Parameter setting:

$$\eta = \sqrt{\bar{n}_{\text{in}}}, \quad \nu = \bar{n}_{\text{in}}^{1/4} + 1, \quad \varepsilon_S = \frac{\varepsilon^2}{2592\, mz(\bar{n}+1)(\bar{n}_{\text{in}}+1)^{1/4}}, \quad \varepsilon_r = \frac{\varepsilon}{2\sqrt{2}\sqrt{z^2\bar{n}+1}}.$$

Query complexity bounds:

$$\boxed{\begin{aligned} N_S &= \Theta\left(\frac{m^3 z^8 (\bar{n}+1)^2 (\bar{n}_{\text{in}}+1)^{1/2}\left(\sqrt{m}+\sqrt{\log(m/\delta)}\right)^2}{\bar{n}_{\text{in}}\varepsilon^4}\right), \\ N_r &= \Theta\left(\frac{(z^2\bar{n}+1)\left(\sqrt{m}+\sqrt{\log(1/\delta)}\right)^2}{(1+\bar{n}_{\text{in}}^{1/4})\varepsilon^2}\right). \end{aligned}}$$

Asymptotic limit:

$$\bar{n}_{\text{in}} \to \infty \quad \Rightarrow \quad N_S, N_r \to 1, \qquad N_{\text{tot}} = (2m+1)N_S + N_r \;\to\; 2m+2.$$

Both learning stages become constant-query in the high-energy limit!

Time Efficiency of the Algorithm

Dominant cost: computation of the symplectic regularization $\tilde{S}$.

Complexity of each component:

- Measurement & displacement reconstruction: $\mathcal{O}(m)$.
- Matrix multiplications: $\mathcal{O}(m^\omega)$ (where ω is the matrix multiplication exponent).
- Principal matrix square root (via Schur decomposition): $\mathcal{O}(m^3)$.

Overall scaling:

$$\text{Total runtime} = \mathcal{O}(m^3) \cdot \text{poly}(z, \bar{n}, \bar{n}_{\text{in}}, 1/\varepsilon).$$

Conclusion:

- The Schur-based root computation dominates the total runtime.
- **Hence, the time complexity scales *asymptotically in the same order* as the query complexity.**

Comparison of Algorithmic Variants

	Symplectic learning	Displacement learning	N_{tot} in high-energy limit
Option A	Vacuum-shared inputs	Two-mode squeezed vacuum	$(2m+1)N_S + N_r \longrightarrow 2m+2$
Option B	Symmetric probes	Two-mode squeezed vacuum	$4mN_S + N_r \longrightarrow 4m+1$
Option C	Vacuum-shared inputs	Single-mode squeezed states	$(2m+1)N_S + 2N_r \longrightarrow 2m+3$
Option D	Symmetric probes	Single-mode squeezed states	$4mN_S + 2N_r \longrightarrow 4m+2$

Observation:

- As $\bar{n}_{\text{in}} \to \infty$, both N_S and N_r approach 1.
- Option A (Vacuum + TMSV) yields the smallest asymptotic query count.

Option A: Full Algorithm Description

Algorithm 1 Learning Gaussian unitaries with auxiliary-system entanglement

Input: Setting of Problem 1, with access to $(2m+1)N_S + N_r$ queries to the unknown Gaussian unitary $G_{\mathbf{r},S}$, where N_S and N_r are specified in Theorem 6.1.

Output: $(\tilde{\mathbf{r}}, \tilde{S})$ such that $D_{\tilde{\mathbf{r}}} U_{\tilde{S}}$ is ε-close to $G_{\mathbf{r},S}$ in energy-constrained diamond norm with probability at least $1-\delta$.

Prepare N_S copies of $G_{\mathbf{r},S} |0\rangle$ and perform heterodyne detection on each of them to construct the mean estimator $\hat{Y}_0$.

for $i \in [2m]$ **do**

  Prepare N_S copies of $G_{\mathbf{r},S} |\eta e_i\rangle$ and perform heterodyne detection on each of them to construct the mean estimator $\hat{Y}_i$.

$\hat{S} \leftarrow [\hat{Y}_1 - \hat{Y}_0, \ldots, \hat{Y}_{2m} - \hat{Y}_0]$

$\tilde{S} \leftarrow (-\Omega \hat{S}^\top \Omega \hat{S})^{-1/2} \hat{S}$

Prepare N_r copies of $U_{S_\nu}^\dagger G_{\mathbf{r},S} U_{\tilde{S}^{-1}} |\nu\rangle^{\otimes m}$ and perform heterodyne detection on each of them to construct the mean estimator $\tilde{\mathbf{m}}$ *(see Section 5.1.1 for an experiment-friendly implementation using only passive optics and input squeezing).*

for $i \in [2m]$ **do**

  $\tilde{\mathbf{r}}_i \leftarrow \tilde{\mathbf{m}}_i / \sqrt{\nu}$

Return: $\tilde{\mathbf{r}}$, $\tilde{S}$

Discussion and open problems

Discussion and Open Problems

Summary.

- We presented the first efficient algorithm for learning bosonic Gaussian unitaries, with both query and time complexity scaling polynomially in the number of modes m.
- The algorithm is experimentally feasible, requiring only heterodyne and homodyne detection, and no online squeezing.
- Error guarantees are given in the energy-constrained diamond norm.
- In the high-energy regime, the query complexity scales linearly in m and becomes independent of squeezing — a "*squeezing independence*" analogous to "energy independence" for Gaussian-state learning. (arXiv:2508.14979)

Techniques. Combination of tools from:

- Continuous-variable quantum information theory,
- Concentration of measure for Gaussian matrices,
- Symplectic matrix regularization under controlled error.

Open Problems (I)

1. **Beyond Unitary Channels.**

   Can general (*non-unitary*) bosonic Gaussian channels be efficiently learned?

   Challenge: establish perturbation bounds for the energy-constrained diamond distance.

2. **Tight Complexity Bounds.**

   Our algorithm achieves $\mathcal{O}(1/\varepsilon^4)$ scaling in target accuracy ε. Can this be improved to $\mathcal{O}(1/\varepsilon^2)$, as in the fermionic case?

Open Problems (II)

3. **Doped Gaussian Unitaries.**
   Explore the trade-off between learning efficiency and the degree of non-Gaussianity. For *t-doped* Gaussian unitaries, state learning is efficient for $t = \mathcal{O}(1)$, but unitary learning remains open. Analogous results for fermionic systems suggest promising directions.

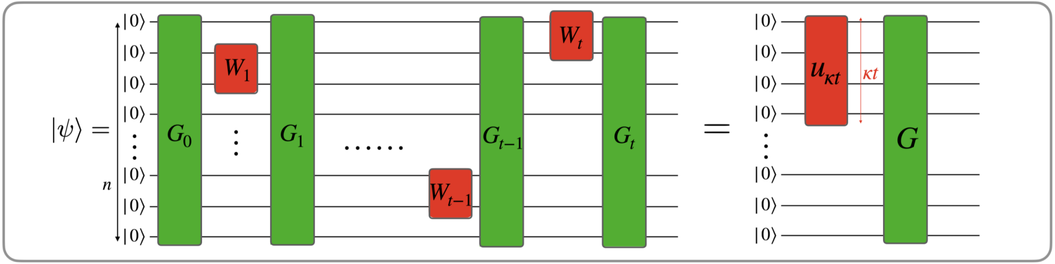

Image source: [Fig. 3, arXiv:2405.01431]

Open Problems (III)

4. **Agnostic Tomography.**

   Finally, the result in this work is proved in the so-called *realizable* setting, where the input is assumed to be exactly a Gaussian unitary. A related open problem is to address the *agnostic* setting: given a bosonic unitary that is ε-close to some Gaussian unitary (in an energy-constrained average-case metric), can we output a Gaussian unitary that is $(\varepsilon + \varepsilon')$-close to the input?

   Here, ε' represents the additional approximation overhead beyond the best approximation achievable within the class of Gaussian unitaries, which arises because the true input need not lie in this class. This problem, known as *agnostic tomography*, has been intensively studied in the discrete-variable setting.

Thank you!

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