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Circuit-to-Hamiltonian from tensor networks and fault tolerance

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Table of Contents

Background

Chapter I

Introduction

01. Introduction

- PCP
- Feynman-Kitaev construction

0) Abstract

We define a map from an arbitrary quantum circuit to a local Hamiltonian whose ground state encodes the quantum computation. All previous maps relied on the Feynman-Kitaev construction, which introduces an ancillary 'clock register' to track the computational steps. Our construction, on the other hand, relies on injective tensor networks with associated parent Hamiltonians, avoiding the introduction of a clock register. This comes at the cost of the ground state containing only a noisy version of the quantum computation, with independent stochastic noise. We can remedy this - making our construction robust - by using quantum fault tolerance. In addition to the stochastic noise, we show that any state with energy density exponentially small in the circuit depth encodes a noisy version of the quantum computation with adversarial noise. We also show that any 'combinatorial state' with energy density polynomially small in depth encodes the quantum computation with adversarial noise. This serves as evidence that any state with energy density polynomially small in depth has a similar property.

As applications, we give a new proof of the QMA-completeness of the local Hamiltonian problem (with logarithmic locality) and show that contracting injective tensor networks to additive error is BQP-hard. We also discuss the implication of our construction to the quantum PCP conjecture, combining with an observation that QMA verification can be done in logarithmic depth.

01. Introduction

0) Abstract

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

01. Introduction

PCP (복잡도)

PCP는 확률적으로 검사할 수 있는 증명(probabilistically checkable proof)을 할 수 있는 판정 문제들의 복잡도 종류이다.

Wikipedia

0-1) PCP(Probabilistically Checkable Proof)

$$\mathrm{PCP}_{c(n),s(n)}[r(n),q(n)]$$

Given a decision problem *L* (or a language L with its alphabet set Σ), a **probabilistically checkable proof system** for *L* with completeness *c*(*n*) and soundness *s*(*n*), where $0 \leq s(n) \leq c(n) \leq 1$, consists of a prover and a verifier. Given a claimed solution x with length n, which might be false, the prover produces a proof *π* which states *x* solves L ($x \in L$, the proof is a string $\in \Sigma^*$). And the verifier is a randomized oracle Turing Machine *V* (the *verifier*) that checks the proof *π* for the statement that *x* solves L (or $x \in L$) and decides whether to accept the statement. The system has the following properties:

- **Completeness**: For any $x \in L$, given the proof *π* produced by the prover of the system, the verifier accepts the statement with probability at least *c*(*n*),
- **Soundness**: For any $x \notin L$, then for any proof *π*, the verifier mistakenly accepts the statement with probability at most *s*(*n*).

The complexity class $\mathrm{PCP}_{c(n),\, s(n)}[r(n),\, q(n)]$ is the class of all decision problems having probabilistically checkable proof systems over binary alphabet of completeness *c*(*n*) and soundness *s*(*n*), where the verifier is non-adaptive, runs in polynomial time, and it has randomness complexity *r*(*n*) and query complexity *q*(*n*).

01. Introduction

$$\mathrm{PCP}_{c(n),s(n)}[r(n), q(n)]$$

0-1) PCP(Probabilistically Checkable Proof)

$$\mathrm{PCP}[\mathrm{poly(n)}, \mathrm{poly(n)}] = \mathrm{NEXP}$$

$$\mathrm{PCP}[O(\log n), O(1)] = \mathrm{NP}$$

The definition of a probabilistically checkable proof was explicitly introduced by Arora and Safra in 1992,[2] although their properties were studied earlier. In 1990 Babai, Fortnow, and Lund proved that **PCP**[poly(*n*), poly(*n*)] = **NEXP**, providing the first nontrivial equivalence between standard proofs (**NEXP**) and probabilistically checkable proofs.[3] The PCP theorem proved in 1992 states that $\mathrm{PCP}[O(\log n), O(1)] = \mathrm{NP}$.[2][4]

- PCP(0, 0) =P
- PCP(다항, 0) = co-RP
- PCP(0, 다항) = NP

주목할 만한 사실:

- PCP(다항, 다항) = NEXP
- NP = PCP(o(log), o(log))이면 NP = P
- NP = PCP(log, 다항)

복잡도 이론이 거둔 큰 성과로 PCP 정리가 있다.

NP = **PCP**(log, O(1)).

01. Introduction

0-2) Feynman-Kitaev construction

$$\mathcal{H} = \mathcal{H}_{\mathrm{clock}} \otimes \mathcal{H}_{\mathrm{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

01. Introduction

0-2) Feynman-Kitaev construction

$$\mathcal{H} = \mathcal{H}_{\mathrm{clock}} \otimes \mathcal{H}_{\mathrm{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

Initial state

$$|\psi_0^0\rangle = |0\rangle_{\mathrm{clock}} \otimes |0\cdots 0\rangle_{\mathrm{data}}$$

Feynman's Hamiltonian

$$H_{\mathrm{F}} = \sum_{t=1}^{N} \left(|t\rangle\langle t-1|_{\mathrm{clock}} \otimes U_t + |t-1\rangle\langle t|_{\mathrm{clock}} \otimes U_t^\dagger \right),$$

Observe also that $H_{\mathrm{F}}|\psi_t^0\rangle = |\psi_{t-1}^0\rangle + |\psi_{t+1}^0\rangle$, so the restriction $H_{\mathrm{F}}\big|_{\mathcal{H}_0}$ is the Hamiltonian of a continuous-time quantum walk on a line [25] of states $|\psi_t^0\rangle$. Using quantum walk techniques, we can show that when we evolve the initial state $|\psi_0^0\rangle$ for a time randomly chosen between 0 and $\Theta\left(N^2\right)$, and measure the clock register, with probability $\Theta\left(N^{-1}\right)$ we will obtain the state $|N\rangle_{\mathrm{clock}}$, and thus $U_N \ldots U_1|0\ldots 0\rangle$ (the result of the circuit U applied to the initial state $|0\ldots 0\rangle$) in the data register. Therefore, evolution with Feynman's Hamiltonian is a universal quantum computer.

Below, in Section II C, we show that Feynman's computer can be built from local terms, by choosing a local implementation of the clock register states and the Hamiltonian terms inducing transitions terms between the states.

01. Introduction

$$\mathcal{H} = \mathcal{H}_{\text{clock}} \otimes \mathcal{H}_{\text{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

0-2) Feynman-Kitaev construction

History state

$$|\psi^{\varphi}_{\text{hist}}\rangle = \frac{1}{\sqrt{N+1}} \sum_{t=0}^{N} |\psi^{\varphi}_t\rangle = \frac{1}{\sqrt{N+1}} \sum_{t=0}^{N} |t\rangle_{\text{clock}} \otimes \underbrace{U_t U_{t-1} \dots U_1 |\varphi\rangle}_{|\varphi_t\rangle_{\text{data}}}$$

Propagation checking Hamiltonian

$$H_{\text{prop}} = \sum_{t=1}^{N} \Big((|t-1\rangle\langle t-1| + |t\rangle\langle t|)_{\text{clock}} \otimes \mathbb{I}_{\text{data}} - |t\rangle\langle t-1|_{\text{clock}} \otimes U_t - |t-1\rangle\langle t|_{\text{clock}} \otimes U_t^{\dagger} \Big)$$

01. Introduction

0-2) Feynman-Kitaev construction

$$\mathcal{H} = \mathcal{H}_{\text{clock}} \otimes \mathcal{H}_{\text{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

Kitaev then used it to give a QMA-complete problem, the *Local Hamiltonian* [17]. He showed how to construct a Hamiltonian with a ground state energy below some bound only if there exists an initial state $|\varphi\rangle$, for which the output qubit of the state $U|\varphi\rangle$ is $|1\rangle$ with high probability. If there is no such state $|\varphi\rangle$, the ground state energy is above some bound. This is one reason behind why determining with high precision the ground state energy of local Hamiltonians is difficult.

Theorem 14.3. *The problem* LOCAL HAMILTONIAN *is* BQNP-*complete with respect to the Karp reduction.*

Classical and quantum computation

$$\mathrm{BQP} \subseteq \mathrm{BQNP} = \mathrm{QMA}$$

01. Introduction

0-2) Feynman-Kitaev construction

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Clock constructions

$$H_{\text{K}} = H_{\text{prop}} + H_{\text{init}} + H_{\text{out}} (+H_{\text{clock}})$$

$$H_{\text{prop}} : \text{propagation}$$

$$H_{\text{init}} = \sum_{\text{ancillas } a} |0\rangle\langle 0|_{\text{clock}} \otimes |1\rangle\langle 1|_a, \quad H_{\text{out}} = |N\rangle\langle N|_{\text{clock}} \otimes |0\rangle\langle 0|_{\text{out}}$$

$$H_{\text{clock}} : \text{clock-checking Hamiltonian}$$

01. Introduction

0-2) Feynman-Kitaev construction

$$\mathcal{H} = \mathcal{H}_{\text{clock}} \otimes \mathcal{H}_{\text{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

The basic building block for Feynman's computer (and Kitaev's Local Hamiltonian construction) is a clock – a register with $N+1$ possible logical states $|0\rangle, \ldots, |N\rangle$, denoting the linear progress of a computation. Originally, Feynman envisioned it being a hopping pointer particle. Here we will look at this construction and other options, their properties, and ways to make them local.

Note that one could also construct clocks with a nonlinear progression of states, without unique forward and backward transitions. In recent quantum complexity results [12], we have seen the combinations of several clock registers, blind alley transitions, railroad-switching paths and path noncommutatitivity, amongst other ideas. However, there are still interesting things to be learned about the basic linear approaches and their relationship to quantum walks, as we will show below.

01. Introduction

$$\mathcal{H} = \mathcal{H}_{\text{clock}} \otimes \mathcal{H}_{\text{data}}$$

Feynman's computer: dynamical construction

Kitaev hamiltonian: static construction

Clock constructions

0-2) Feynman-Kitaev construction

Hopping Hamiltonian

$$H_N^{\text{walk}} = -\sum_{t=0}^{N-1} \left(|t+1\rangle\langle t| + |t\rangle\langle t+1| \right)$$

Laplacian quantum walk

$$H_N^{\text{L}} = \sum_{t=0}^{N-1} \left(|t\rangle - |t+1\rangle \right) \left(\langle t| - \langle t+1| \right)$$

Quantum walk on a line →

- 1-excitation sector of ferromagnetic **XX-model spin** chain
- 1-excitation sector of ferromagnetic **XXX-model spin** chain ✓

01. Introduction

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Hopping Hamiltonian

$$H_N^{\text{walk}} = -\sum_{t=0}^{N-1} \left(|t+1\rangle\langle t| + |t\rangle\langle t+1|\right)$$

Laplacian quantum walk

$$H_N^{\text{L}} = \sum_{t=0}^{N-1} \left(|t\rangle - |t+1\rangle\right)\left(\langle t| - \langle t+1|\right)$$

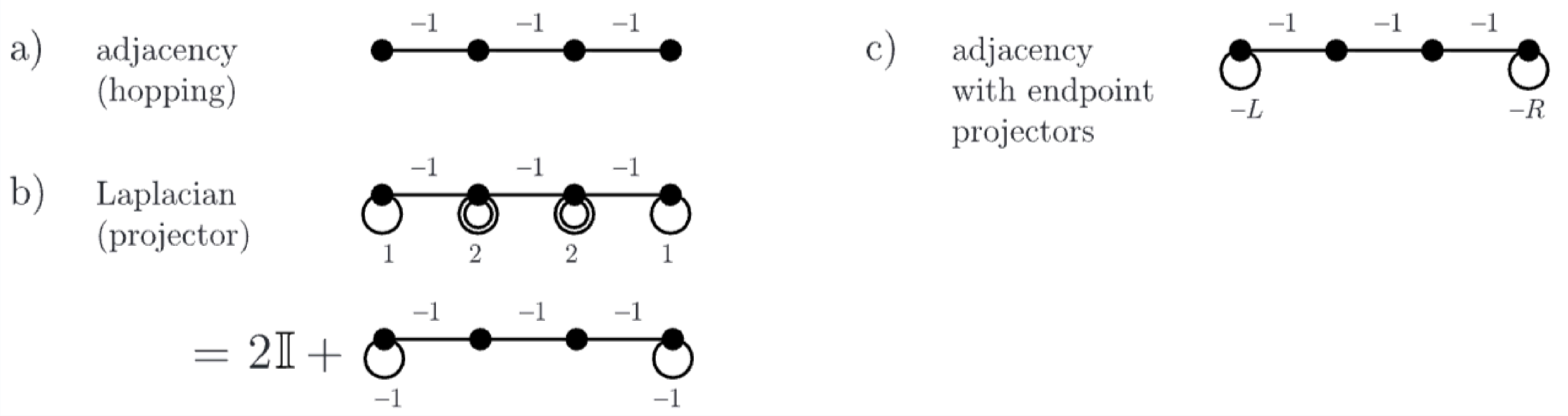

FIG. 1. Quantum walks on a line of length N. a) The quantum walk on a line (hopping) Hamiltonian H_N^{walk} (8) is the negative of the adjacency matrix. b) The Laplacian walk H_N^{L} (9) includes a self loop on each vertex for each outgoing edge. c) A more general version $H_N^{(L,R)}$ (21) parametrized by a pair L, R includes endpoint projectors (loops) $-L|0\rangle\langle 0|$, $-R|N\rangle\langle N|$.

01. Introduction

0-2) Feynman-Kitaev construction

Clocks in Feynman's computer and Kitaev's local Hamiltonian: Bias, gaps, idling, and pulse tuning

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Finally, in the *yes* case, the history state for a good witness accepted with probability $\geq 1-\epsilon$ has energy at most $\frac{\epsilon}{N}$, for our choice $\epsilon = \frac{1}{N^2}$. Altogether, the lowest eigenvalue in the *yes* and *no* cases are

$$E_{yes} \leq \frac{\epsilon}{N} \leq \frac{1}{N^3}, \qquad E_{no} \geq \frac{const.}{N^2}. \tag{59}$$

Thus for a circuit amplified to soundness at most $\epsilon = O\left(N^{-2}\right)$ and completeness at least $1-\epsilon$, we obtain a new promise gap $E_{no} - E_{yes} = \Omega\left(N^{-2}\right)$. □

01. Introduction

0-2) Feynman-Kitaev construction

Detailed Analysis of Circuit-to-Hamiltonian Mappings

James D. Watson

Theorem 3.1. *The ground state energy of a standard-form Hamiltonian, $H_{QMA} \in \mathcal{B}(\mathbb{C}^d)^{\otimes n}$, encoding the verification computation of a QMA instance with total run-time $T = \text{poly}(n)$ is bounded as*

$$0 \leq \lambda_0\left(H_{QMA}^{(YES)}\right) \leq e^{-O(\text{poly}(n))} \quad (3.1)$$

$$1 - \cos\left(\frac{\pi}{2T}\right) - e^{-O(\text{poly}(n))} \leq \lambda_0\left(H_{QMA}^{(NO)}\right) \leq 1 - \cos\left(\frac{\pi}{2T}\right). \quad (3.2)$$

01. Introduction

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

1) Connection to quantum PCP conjecture

Connection to quantum PCP conjecture: The quantum PCP conjecture [13, 14] states that it is QMA-hard to decide if the ground energy density of a local Hamiltonian problem is less than a given number a or more than $a+\Delta$ for a constant Δ. A 'polylog weaker' version of this conjecture - QMA hardness of deciding that ground energy density is $\leq a$ or $> a + \frac{1}{\text{polylog} n}$ - is also open (even when the locality is relaxed to be polylog n). In the equivalent formulation in terms of probabilistic proof checking [14], this polylog-weaker quantum PCP conjecture is expressed as the (presumed) inclusion QMA $\stackrel{?}{\subseteq}$ QPCP[polylog]. See Appendix B for a discussion on known soundness results.

$$\mathrm{QMA} \subseteq^{?} \mathrm{QPCP}[\mathrm{polylog}]$$

Feynman-Kitaev(QMA) → QPCP

↘ PEPS(Projected Entangled Pair States)
→ injective TN
Adversarial noise model ↗

01. Introduction

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
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1) Connection to quantum PCP conjecture

	Feynman-Kitaev construction [1]	**Present construction**
Ground state	Superposition over partial computations of W	Tensor network encoding a noisy version of W with i.i.d noise per wire
Low-energy states	States with energy density $\frac{O(1)}{\lvert W\rvert^3}$ encode W	Combinatorial states with $\frac{O(1)}{D}$ fraction violations encode a noisy version of W with adversarial noise (Theorem 4.4). • States with energy density $e^{-\Omega(D\log D)}$ (for $D = O(\log\lvert W\rvert)$) encode a noisy version of W with adversarial noise (Theorem 4.3).
Limitation	There exists a combinatorial state with $\frac{O(1)}{\lvert W\rvert}$ fraction of violations containing no information about W (see Remark 5.2).	There exists a combinatorial state with $\frac{O(1)}{D}$ fraction of violations contain no information about W.

Table 1: A comparison between the Feynman-Kitaev mapping and our construction for quantum circuit W of depth D. Our main open question is that any state with energy density $\frac{1}{\mathrm{poly}(D)}$ encode noisy version of W with adversarial noise. Since we can choose $D = O(\log\lvert W\rvert)$ in QMA protocols (Section 5 and Appendix D), this serves as a link between polylog quantum PCP and adversarial quantum fault tolerance.

01. Introduction

- Connection to quantum PCP conjecture
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1) Connection to quantum PCP conjecture

The details of the construction appear in Section 2, where we use standard teleportation instead of measurement-based quantum computing. A high level overview is as follows, using a simple circuit $U_2U_1|0\rangle$ involving 1 qubit gates on $|0\rangle$. Introduce 5 qubits in the state $|0\rangle \otimes (\mathrm{I} \otimes U_1)|\Phi_\mathrm{I}\rangle \otimes (\mathrm{I} \otimes U_2)|\Phi_\mathrm{I}\rangle$, where $|\Phi_\mathrm{I}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Projecting qubits 1, 2 and 3, 4 with $|\Phi_\mathrm{I}\rangle\langle\Phi_\mathrm{I}|$ would lead to the desired state $U_2U_1|0\rangle$ on qubit 5. This procedure defines a tensor network state known as PEPS [16]. However, this tensor network

Our main technical contribution is a characterization of low-energy states of the parent Hamiltonian as adversarial computations of the quantum circuit. In particular, we consider an adversarial noise model where, in each layer of the circuit, a certain fraction of qubits are deviated arbitrarily by the adversary. Consider a quantum circuit W of depth D (that may be, for examples, a QMA verification circuit or a BQP circuit) and let its parent Hamiltonian be $H_\mathrm{parent} = \sum_{i=1}^m h_i$ (assume here $0 \le h_i \le 1$ for simplicity). We exhibit the following properties for low-energy states of H_parent:

- For a circuit of depth $D = O(\log|W|)$, any state $|\psi\rangle$ with energy density $e^{-\Omega(D\log D)}$, i.e.,

$$\frac{1}{m}\langle\psi| H_\mathrm{parent}|\psi\rangle \le e^{-\Omega(D\log D)},$$

  can be viewed as the output of the circuit with $O(\delta^2)$ fraction of adversarial noise per layer. See Section 4.3 for the precise statement and proof.

- Any combinatorial state with energy density (equal to the fraction of violated constraints) $\frac{1}{\mathrm{poly}(D)}$, i.e.,

$$\frac{1}{m}|\{i : \langle\psi| h_i |\psi\rangle \neq 0\}| \le \frac{1}{\mathrm{poly}(D)},$$

  can be viewed as the output of the circuit with $O(\delta^2)$ fraction of adversarial noise per layer. See Section 4.2 for the precise statement and proof.

- MBQC(measure-based)

→ Standard teleportation

→ PEPS(tensor networks)

- Adversarial noise model

01. Introduction

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

2) New proof of QMA-completeness of local Hamiltonian

New proof of QMA-completeness of local Hamiltonian: The first application of quantum circuit-to-Hamiltonian mapping was Kitaev's proof of the QMA-completeness of the local Hamiltonian problem. In Section 6, we apply our construction to give a new proof of this seminal result for the case of logarithmic-local Hamiltonians. In particular, we prove that determining if the ground energy density of a $O(\log n)$-local Hamiltonian family is less than a given number a or more than $a + \frac{1}{\mathrm{poly}(n)}$ is QMA-complete. While we have not been able to prove the same statement for the constant locality case (see discussion below), we remark that our proof is completely independent of the Feynman-Kitaev clock construction.

Theorem 14.3. *The problem* LOCAL HAMILTONIAN *is* BQNP*-complete with respect to the Karp reduction.*

Logarithmic-local Hamiltonian (no clock)

01. Introduction

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

3) Complexity of injective tensor networks

Complexity of injective tensor networks: Injective tensor networks constitute a more physical family of quantum states and have been shown to be efficiently preparable on a quantum computer [21, 22] and contractable in classical quasi-polynomial time [23] under assumptions on the parent Hamiltonian spectral gap. However, the lack of the postselection ability makes it less clear how to characterize injective TN from a complexity-theoretic point of view.

Combining our construction with existing quantum fault-tolerance schemes for local stochastic noise [24], we conclude that preparing injective TN states on a quantum computer is BQP-hard. This can be seen as a complement to prior works [21, 22], that showed preparing injective TN states under spectral gap assumptions is in BQP. Compared with the PostBQP-hardness shown in [15], the BQP-hardness naturally reflects the non-postselecting nature of injective TN. Regarding the classical complexity of injective TN, our construction also implies that evaluating local observable expectation values on injective-TN states is BQP-hard to $O(1)$-additive error. In addition, we show the same task for a non-local observable is #P-hard to $O(1)$-multiplicative error.

- Preparing injective TN states → BQP-hard

Chapter II

Model (injective-TN)

02. Model

Let the EPR states be $|\Phi_I\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\Phi_X\rangle = (\mathrm{I} \otimes X)|\Phi_I\rangle, |\Phi_{XZ}\rangle = (\mathrm{I} \otimes XZ)|\Phi_I\rangle, |\Phi_Z\rangle = (\mathrm{I} \otimes Z)|\Phi_I\rangle$. Denote $\mathcal{P} = \{I, X, XZ, Z\}$. For an operator A in a Hilbert space with tensor product structure $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, we denote by $\mathrm{supp}(A)$ the span of eigenvectors of A with nonzero eigenvalues and by $\mathrm{loc}(A)$ the set of subsystems on which A acts nontrivially.

0) Notation

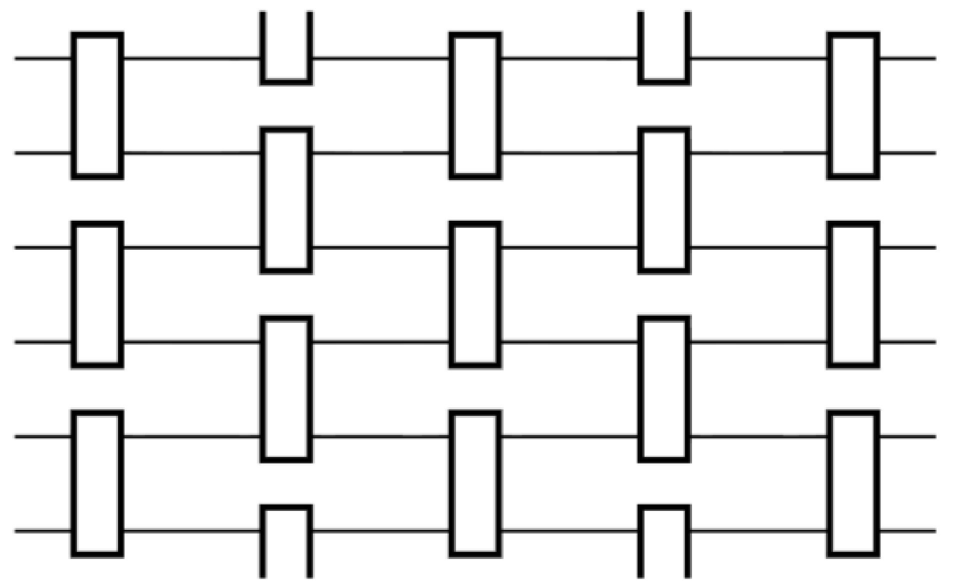
$$|\Phi_I\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_X\rangle = (I \otimes X)|\Phi_I\rangle$$
$$|\Phi_{XZ}\rangle = (I \otimes XZ)|\Phi_I\rangle$$
$$|\Phi_Z\rangle = (I \otimes Z)|\Phi_I\rangle$$

$$\mathcal{P} = \{I, X, XZ, Z\}$$

02. Model

1) Quantum circuit to tensor network

$$|\Phi_I\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \mathcal{P} = \{I, X, XZ, Z\}$$
$$|\Phi_X\rangle = (I \otimes X)|\Phi_I\rangle$$
$$|\Phi_{XZ}\rangle = (I \otimes XZ)|\Phi_I\rangle$$
$$|\Phi_Z\rangle = (I \otimes Z)|\Phi_I\rangle$$



$$\left|\Phi_{p,q}^{(\ell)}\right\rangle = \frac{\left[I_{1,2} \otimes \left(U_{p,q}^{(\ell)}\right)_{3,4}\right](|00\rangle_{1,3} + |11\rangle_{1,3}) \otimes (|00\rangle_{2,4} + |11\rangle_{2,4})}{2}$$

Choi state

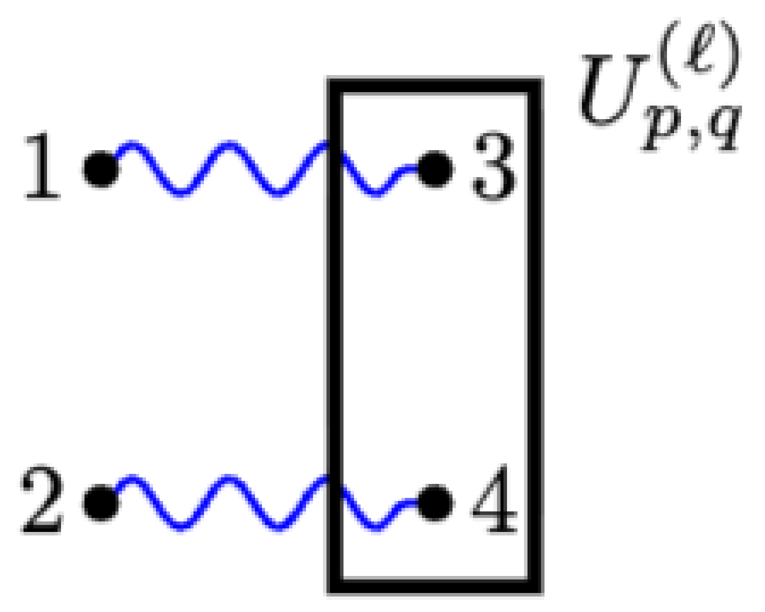


Figure 2: Representation of the state $\left|\Phi_{p,q}^{(\ell)}\right\rangle$. Qubits 1 and 3, as well as 2 and 4, are in the Bell state $|\Phi_I\rangle$, which is indicated by the blue wavy lines. The unitary $U_{p,q}^{(\ell)}$ is applied to qubits 3 and 4 (black box).

02. Model

1) Quantum circuit to tensor network

$$|\Phi_I\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \mathcal{P} = \{I, X, XZ, Z\}$$

$$|\Phi_X\rangle = (I \otimes X)|\Phi_I\rangle$$

$$|\Phi_{XZ}\rangle = (I \otimes XZ)|\Phi_I\rangle$$

$$|\Phi_Z\rangle = (I \otimes Z)|\Phi_I\rangle$$

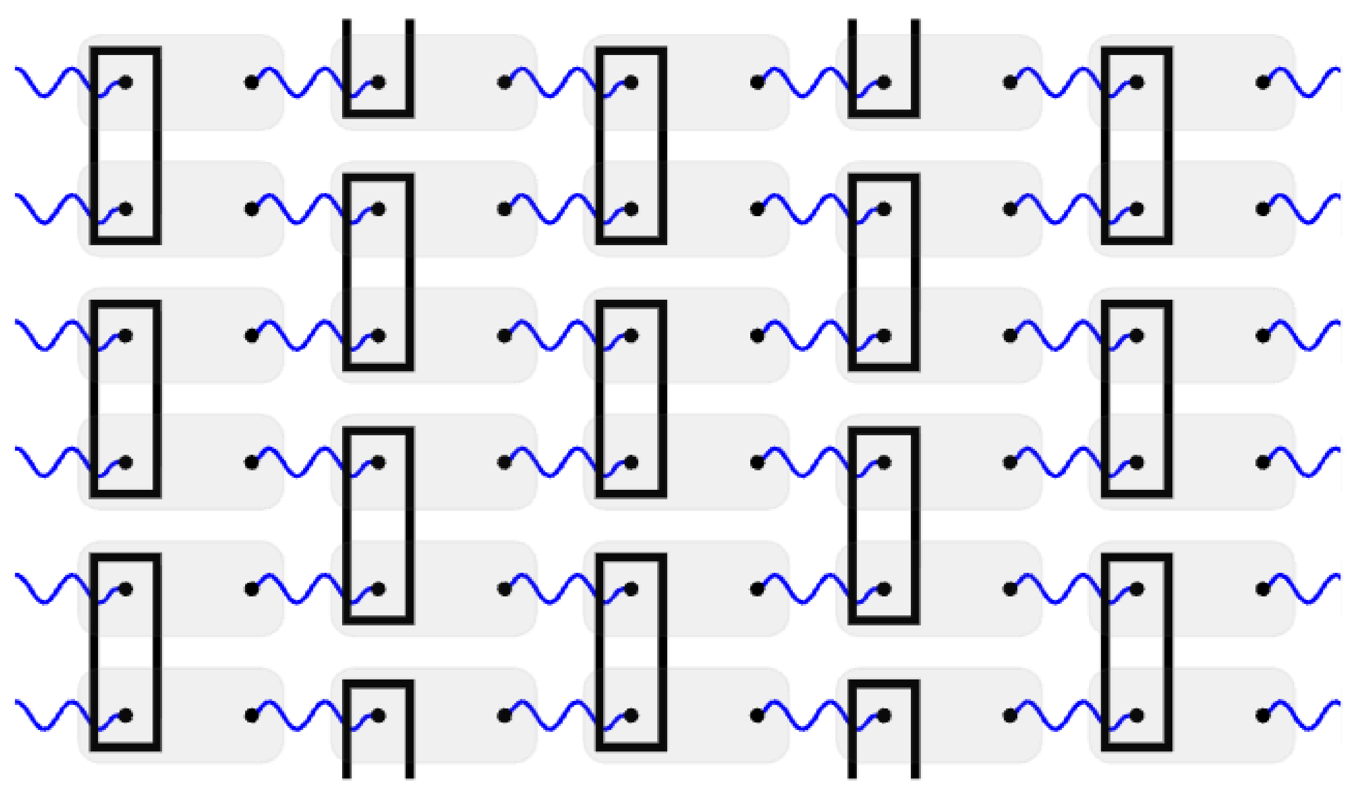
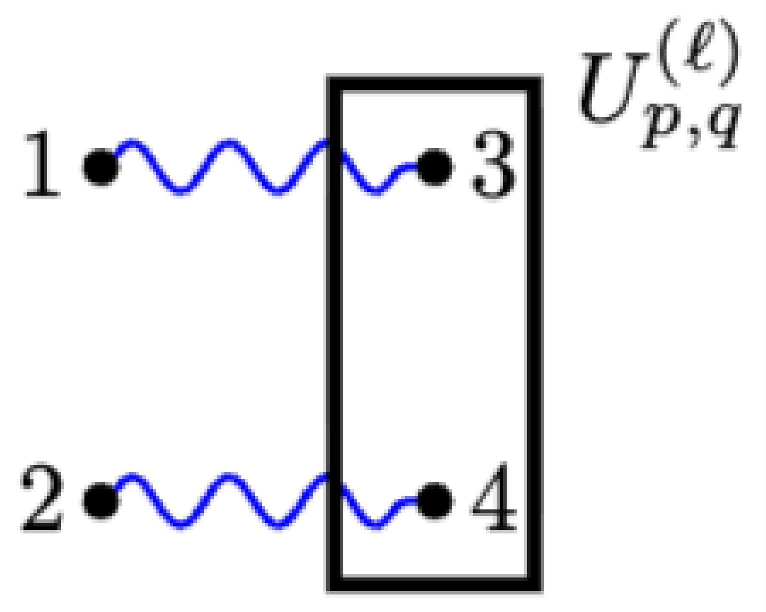


Figure 3: The circuit W (Figure 1) converted into a tensor network. We introduce a Bell pair for every position in the circuit (black dots connected by a wavy line) and apply the unitary operation corresponding to the location in the circuit (cf. Figure 2). We then apply projectors on pairs of qubits (gray boxes).



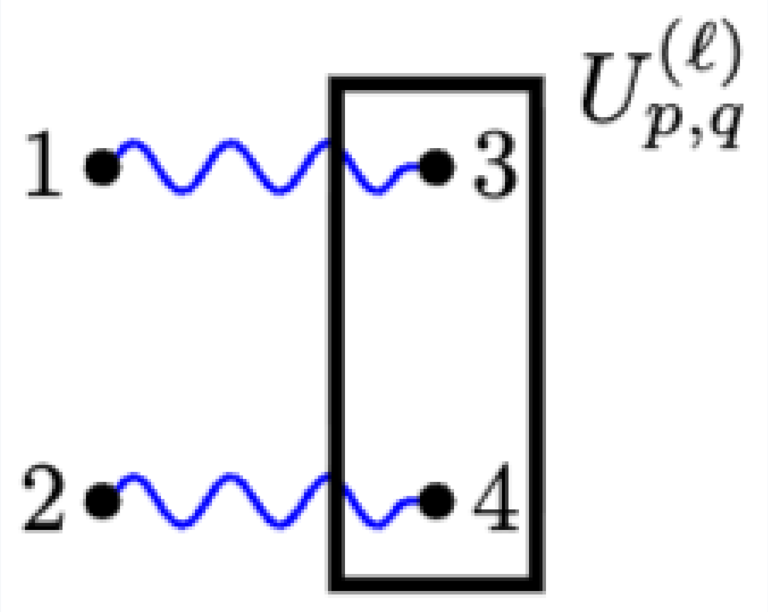
$$\left|\Phi_{p,q}^{(\ell)}\right\rangle = \frac{\left[I_{1,2} \otimes \left(U_{p,q}^{(\ell)}\right)_{3,4}\right](|00\rangle_{1,3} + |11\rangle_{1,3}) \otimes (|00\rangle_{2,4} + |11\rangle_{2,4})}{2}$$

02. Model

$$|\Phi_I\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \mathcal{P} = \{I, X, XZ, Z\}$$

$$|\Phi_X\rangle = (I \otimes X)|\Phi_I\rangle$$
$$|\Phi_{XZ}\rangle = (I \otimes XZ)|\Phi_I\rangle$$
$$|\Phi_Z\rangle = (I \otimes Z)|\Phi_I\rangle$$

1) Quantum circuit to tensor network



$$|\Phi_{W,\xi}\rangle = |\xi\rangle \otimes \bigotimes_{\ell,p,q} \left|\Phi_{p,q}^{(\ell)}\right\rangle$$

$$\left|\Phi_{p,q}^{(\ell)}\right\rangle = \frac{\left[I_{1,2} \otimes \left(U_{p,q}^{(\ell)}\right)_{3,4}\right] (|00\rangle_{1,3} + |11\rangle_{1,3}) \otimes (|00\rangle_{2,4} + |11\rangle_{2,4})}{2}$$

Total # of qubits of TN: (2D+1)n

$$|\xi\rangle = |0\rangle^{\otimes n} \quad \Pi_W \stackrel{\Delta}{=} \bigotimes_{l,p,q} P_{p,q}^{(l)} \qquad P = |\Phi_I\rangle^{\otimes 2}\langle\Phi_I|^{\otimes 2}$$

$$\Pi_W|\Phi_{W,\xi}\rangle \propto |\Phi_I\rangle^{\otimes nD} \otimes \left(W|0\rangle^{\otimes n}\right)$$

→ PEPS
(projected entangled pair states)

02. Model

$$\Pi_W|\Phi_{W,\xi}\rangle \propto |\Phi_I\rangle^{\otimes nD} \otimes \left(W|0\rangle^{\otimes n}\right)$$

→ PEPS
(projected entangled pair states)

1) Quantum circuit to tensor network

δ - perturbation

$$Q = |\Phi_I\rangle\langle\Phi_I| + \delta \sum_{P\in\{X,XZ,Z\}} |\Phi_P\rangle\langle\Phi_P|$$

$$|\Phi_{W,\xi}\rangle = |\xi\rangle \otimes \bigotimes_{\ell\in[D],p,q} \left|\Phi^{(\ell)}_{p,q}\right\rangle$$

We say that a tensor network is δ-*injective* when its local maps have singular values lower bounded by δ. The tensor network defined in the previous section is non-injective since the projectors are singular. To make the tensor network injective, we follow the procedure in [25] and replace the projectors P by a δ-perturbation

$$|\Psi_{W,\xi}\rangle \triangleq Q^{\otimes nD}\,|\Phi_{W,\xi}\rangle$$

→ injective PEPS

02. Model

$$|\Psi_{W,\xi}\rangle \triangleq Q^{\otimes nD}|\Phi_{W,\xi}\rangle$$

→ injective PEPS

1) Quantum circuit to tensor network

We introduce several notations. Let T be the number of gates, let $|\Phi_{\vec{P}}\rangle = \bigotimes_{i=1}^{T}|\Phi_{P_i}\rangle$ for $\vec{P} \in \mathcal{P}^{\otimes T}$ and let $|\vec{P}|$ denote the number of nontrivial operators in $\vec{P}$. Let $W_\ell = \otimes_{i\in \text{ layer } \ell}U_i$ and $\tilde{P}_\ell = \otimes_{i\in \text{ layer } \ell}P_i$ be the unitaries and the errors in the ℓ-th layer of W. Abusing notation, we sometimes denote $U_i \in W_\ell$ and $P_i \in \tilde{P}_\ell$ to mean that the unitaries and Pauli errors are in layer ℓ.

The key observation is that the injective tensor network represents a noisy version of the quantum computation.

$$|\Psi_{W,\xi}\rangle = \sum_{\vec{P}\in\mathcal{P}^{\otimes T}} \delta^{|\vec{P}|}|\Phi_{\vec{P}}\rangle\langle\Phi_{\vec{P}}|\Phi_{W,\xi}\rangle$$

let unitary V

$$V = \sum_{\vec{P}\in\mathcal{P}^{\otimes nD}} |\Phi_{\vec{P}}\rangle\langle\Phi_{\vec{P}}| \otimes (W_D\tilde{P}_D \dots W_1\tilde{P}_1)$$

s.t.

$$V^\dagger|\Psi_{W,\xi}\rangle \propto \sum_{\vec{P}\in\mathcal{P}^{\otimes T}} \delta^{|\vec{P}|}|\Phi_{\vec{P}}\rangle \otimes |\xi\rangle$$

02. Model

$$|\Psi_{W,\xi}\rangle \triangleq Q^{\otimes nD} |\Phi_{W,\xi}\rangle$$

→ injective PEPS

1) Quantum circuit to tensor network

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s.t.

$$V^\dagger|\Psi_{W,\xi}\rangle \propto \sum_{\vec{P}\in\mathcal{P}^{\otimes T}} \delta^{|\vec{P}|}|\Phi_{\vec{P}}\rangle \otimes |\xi\rangle$$

Note that the state $\sum_{\vec{P}\in\mathcal{P}^{\otimes T}} \delta^{|\vec{P}|}|\Phi_{\vec{P}}\rangle$ is a simple product state since the noise is i.i.d local. Thus, when $|\xi\rangle = |0\rangle^{\otimes n}$ (which arises for computations in BQP), the state $|\Psi_{W,\xi}\rangle$ can be prepared by a quantum circuit. This is similar to the Feynman-Kitaev history state [1], which can be prepared efficiently for quantum computations in BQP.

In general, we are not restricted to choosing the same injectivity parameter δ across the whole circuit. In fact, some of our later results are proved by varying δ between locations in the circuit.

Independent and Identical Distributed random varibles

02. Model

$$|\Psi_{W,\xi}\rangle \triangleq Q^{\otimes nD}\,|\Phi_{W,\xi}\rangle$$

→ injective PEPS

2) Parent Hamiltonian

The nice property of the injective tensor network state $|\Psi_{W,\xi}\rangle$ is that it is the unique ground state of a local Hamiltonian. In particular, we consider the $n(2D+1)$-qubit Hilbert space containing the PEPS state $|\Psi_{W,\xi}\rangle$ corresponding to a circuit W.

Let $\Lambda = \delta\,|\Phi_I\rangle\langle\Phi_I| + \sum_{p\in\{X,XZ,Z\}} |\Phi_p\rangle\langle\Phi_p|$, such that $Q \propto \Lambda^{-1}$.

Definition 2.2 (Parent Hamiltonian)**.** *Associate for each gate two-qubit gate U in the circuit an 8-qubit Hamiltonian term $h_U = \Lambda^{\otimes 4}(\mathrm{I} - |\Phi_U\rangle\langle\Phi_U|)\Lambda^{\otimes 4}$. Furthermore, suppose the initial state $|\xi\rangle$ is the unique ground state of a frustration-free local Hamiltonian $H_\xi = \sum_j g_j$. Then the unnormalized state $\Phi_{W,\xi}$ is the unique ground state of the frustration-free Hamiltonian $H_{\text{parent}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} + \sum_{U\in W} h_U$, where $N(j)$ is the set of EPR locations that have intersecting support with g_j (see Figure 4). We refer to the first set of terms as H_{in} (input) and the second set as H_{prop} (propagation).*

$$H_{parent} = H_{in} + H_{prop}$$

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U\in W} h_U$$

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

2) Parent Hamiltonian

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U \in W} h_U$$

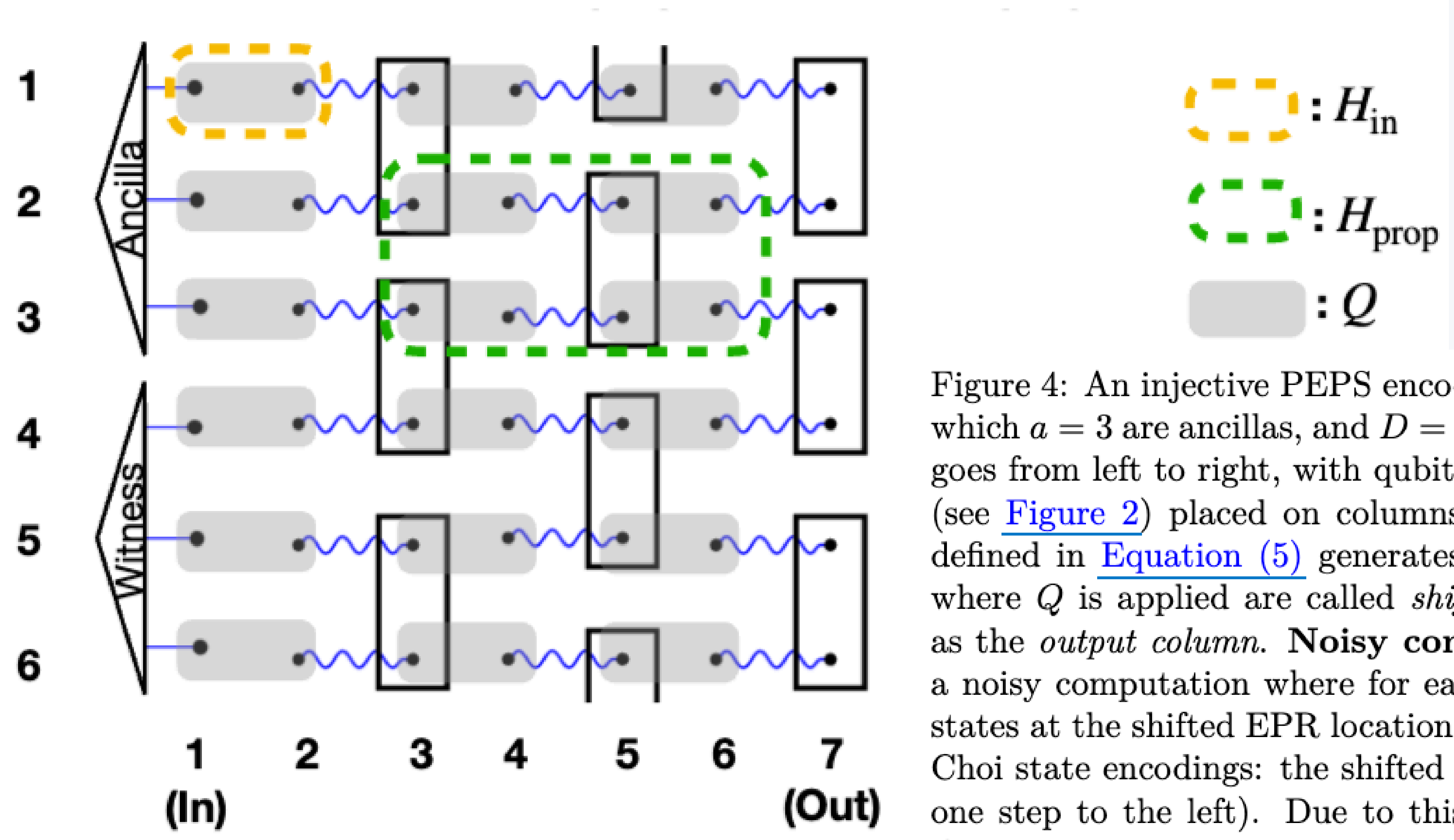

Figure 4: An injective PEPS encoding noisy quantum computation shown with $n = 6$ qubits (black dots), of which $a = 3$ are ancillas, and $D = 3$ layers of two-qubit gates in the brickwork architecture. The computation goes from left to right, with qubits on column 1 being the input. **Gates** are encoded in 4-qubit Choi states (see Figure 2) placed on columns (2,3), (4,5), and so on. Applying the invertible map Q (gray box) as defined in Equation (5) generates a noisy computation on the last column (indexed 7). The qubit pairs where Q is applied are called *shifted* EPR locations. We refer to the last column of qubits in the PEPS as the *output column*. **Noisy computation**: After Q is applied, the output column can be interpreted as a noisy computation where for each layer of the circuit, the present noise pattern is specified by the EPR states at the shifted EPR locations (the word 'shifted' is to avoid confusion with the original locations of the Choi state encodings: the shifted EPR locations are the same as shifting the original Choi state's locations one step to the left). Due to this correspondence, we refer to the first two columns (indexed 1,2) as the *first layer*, the next two columns (indexed 3,4) as the *second layer*, and so on. **Parent Hamiltonian**: A propagation term (dashed green) acts on 8 qubits, while an initialization term (dashed yellow) acts on the first 2 qubits and only on each ancilla row (indexed 1,2,3).

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U \in W} h_U$$

2) Local indistinguishability

A conceptual challenge that any circuit-to-Hamiltonian construction must resolve is local indistinguishability. As discussed in [26], the argument is as follows. Consider a n-qubit quantum state $|\psi\rangle$ that is subject to either I_n unitary or the Z_n unitary on the last qubit. It is possible that $|\psi\rangle$ and $Z_n |\psi\rangle$ are locally indistinguishable (such as in the context of CAT states). But then how can a local constraint detect the difference between the two actions of unitaries? The Feynman-Kitaev approach solves this problem by using the clock register - see [26]. ✓

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U \in W} h_U$$

2) Local indistinguishability

In our context, noise plays a crucial role in handling the local indistinguishability issue and showing that local changes can be locally detected. Indeed, consider h_{I_n} and h_{Z_n} as the two tensor network Hamiltonian terms corresponding to the two possible gates. Let π_{I_n} and π_{Z_n} be their respective ground spaces. If $|\psi\rangle$ was subject to the I_n unitary, the corresponding tensor network state $|\Psi\rangle$ would be in the support of π_{I_n}. Thus, we can lower bound the energy of $|\Psi\rangle$ with respect to h_{Z_n} by upper bounding ✓

$$\|\pi_{Z_n} |\Psi\rangle \| = \|\pi_{Z_n}\pi_{\mathrm{I}_n} |\Psi\rangle \| \leq \|\pi_{Z_n}\pi_{\mathrm{I}_n}\|.$$

We argue that $\|\pi_{Z_n}\pi_{\mathrm{I}_n}\| \leq 1 - \delta^6/2$ when $\delta < \frac{1}{4}$. For this, we will show that the overlap between any two vectors from the two subspaces is $\leq 1 - \frac{\delta^6}{2}$.

We have the following characterization of the ground space of the propagation term h_U for general one-qubit gate U (the two-qubit case can be similarly derived).

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

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2) Local indistinguishability

Claim 2.3. *A general vector in the ground space of* $h_U = \Lambda^{\otimes 2}(\mathrm{I} - |\Phi_U\rangle\langle\Phi_U|)\Lambda^{\otimes 2}$ *can be written as* $\sum_{\vec{P}=(P_1,P_2)\in\mathcal{P}^{\otimes 2}} \delta^{|\vec{P}|}\,\mathrm{Tr}[MP_1UP_2]\,|\Phi_{\vec{P}}\rangle$, *for arbitrary one-qubit operator* M.

Proof. The ground space of $(\Lambda^{-1})^{\otimes 2} h_U (\Lambda^{-1})^{\otimes 2}$ is $\mathrm{span}\{|\psi_1\rangle|\Phi_U\rangle|\psi_2\rangle : \forall |\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2\}$. Thus the ground space of h_U is

$$\begin{aligned} &\mathrm{span}_{|\psi_1\rangle,|\psi_2\rangle}\{\Lambda^{\otimes 2}|\psi_1\rangle|\Phi_U\rangle|\psi_2\rangle\} \\ &= \mathrm{span}_{|\psi_1\rangle,|\psi_2\rangle}\left\{\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \langle\psi_1|P_1UP_2|\psi_2\rangle|\Phi_{\vec{P}}\rangle\right\} \\ &= \mathrm{span}_{M\in\mathbb{C}^{2\times 2}}\left\{\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \mathrm{Tr}[MP_1UP_2]|\Phi_{\vec{P}}\rangle\right\}. \end{aligned}$$

Applying the above claim, a general state in π_{I_n} can be written as $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{|\vec{P}|}\,\mathrm{Tr}\{MP_1P_2\}\,|\Phi_{\vec{P}}\rangle$ for arbitrary operator M subject to the normalization condition $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|}|\mathrm{Tr}(MP_1P_2)|^2 = 1$. Sim ilarly, a general state in π_{Z_n} can be written as $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{|\vec{P}|}\,\mathrm{Tr}\{ZNP_1ZP_2\}\,|\Phi_{\vec{P}}\rangle$, for arbitrary opera tor N (we add a Pauli Z in front of N for later convenience) subject to the normalization conditio $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|}|\mathrm{Tr}(NP_1P_2)|^2 = 1$. It is clear that the two vectors must be different - no matrices M, N satisfie $\mathrm{Tr}(MP_1P_2) = \mathrm{Tr}(ZNP_1ZP_2) = (-1)^{\mathrm{Ind}(P_2\in X,Y)}\,\mathrm{Tr}(NP_1P_2)$ for all Paulis P_1, P_2, where Ind denotes the indi cator variable. Otherwise, $M = (-1)^{\mathrm{Ind}(P_2\in X,Y)}N$ for all P_2, which forces $M = 0$. We can also get a quantita tive bound by arguing that the ℓ_1 distance $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|}|\mathrm{Tr}\big((M - (-1)^{\mathrm{Ind}(P_2\in X,Y)}N)P_1P_2\big)|^2$ must be $\geq \delta$ For contradiction, suppose the opposite holds. Then for all P_1, P_2, $|\mathrm{Tr}\big((M - (-1)^{\mathrm{Ind}(P_2\in X,Y)}N)P_1P_2\big)| \leq \delta$ For a fixed P_1P_2, we can choose P_1, P_2 such that $P_2 \in \{\mathrm{I}, Z\}$ as well as $P_2 \in \{X, Y\}$. This means we have

$$|\mathrm{Tr}((M \pm N)P_1P_2)| \leq \delta, \implies |\mathrm{Tr}(MP_1P_2)| \leq \delta.$$

The implication uses triangle inequality. This forces the normalization condition to be

$$\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|}|\mathrm{Tr}(MP_1P_2)|^2 \leq 16\delta^2 < 1,$$

a contradiction.

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U \in W} h_U$$

2) Local indistinguishability

Applying the above claim, a general state in π_{I_n} can be written as $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{|\vec{P}|} \operatorname{Tr}\{MP_1P_2\} |\Phi_{\vec{P}}\rangle$, for arbitrary operator M subject to the normalization condition $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|} |\operatorname{Tr}(MP_1P_2)|^2 = 1$. Similarly, a general state in π_{Z_n} can be written as $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{|\vec{P}|} \operatorname{Tr}\{ZNP_1ZP_2\} |\Phi_{\vec{P}}\rangle$, for arbitrary operator N (we add a Pauli Z in front of N for later convenience) subject to the normalization condition $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|} |\operatorname{Tr}(NP_1P_2)|^2 = 1$. It is clear that the two vectors must be different - no matrices M, N satisfies $\operatorname{Tr}(MP_1P_2) = \operatorname{Tr}(ZNP_1ZP_2) = (-1)^{\operatorname{Ind}(P_2\in X,Y)} \operatorname{Tr}(NP_1P_2)$ for all Paulis P_1, P_2, where Ind denotes the indicator variable. Otherwise, $M = (-1)^{\operatorname{Ind}(P_2\in X,Y)} N$ for all P_2, which forces $M = 0$. We can also get a quantitative bound by arguing that the ℓ_1 distance $\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|} |\operatorname{Tr}\big((M - (-1)^{\operatorname{Ind}(P_2\in X,Y)} N)P_1P_2\big)|^2$ must be $\geq \delta^6$. For contradiction, suppose the opposite holds. Then for all P_1, P_2, $|\operatorname{Tr}\big((M - (-1)^{\operatorname{Ind}(P_2\in X,Y)} N)P_1P_2\big)| \leq \delta$. For a fixed P_1P_2, we can choose P_1, P_2 such that $P_2 \in \{\mathrm{I}, Z\}$ as well as $P_2 \in \{X, Y\}$. This means we have

$$|\operatorname{Tr}((M \pm N)P_1P_2)| \leq \delta, \implies |\operatorname{Tr}(MP_1P_2)| \leq \delta.$$

The implication uses triangle inequality. This forces the normalization condition to be

$$\sum_{\vec{P}\in\mathcal{P}^{\otimes 2}} \delta^{2|\vec{P}|} |\operatorname{Tr}(MP_1P_2)|^2 \leq 16\delta^2 < 1,$$

a contradiction.

02. Model

$$H_{parent} = H_{in} + H_{prop}$$

$$H_{\text{in}} = \sum_j \Lambda^{\otimes N(j)} g_j \Lambda^{\otimes N(j)} \qquad H_{\text{prop}} = \sum_{U \in W} h_U$$

2) Connection with prior works

A scheme related to ours is that of Ref. [27] in which the authors give a construction of quantum error-correcting subsystem codes with almost linear distance. Their construction can be understood as a map from fault-tolerant Clifford circuits that facilitate check measurements to a set of non-commuting Pauli-check operators. More concretely, each location in the circuit is associated with a qubit and each Clifford gate is associated with a Pauli operator that stabilizes the gate. For example, the idling gate (wire) is stabilized by XX and ZZ operating on the in- and out-locations. The main difference with our setting is that we do not need to assume Clifford circuit. Furthermore, our Hamiltonian remains frustration-free, whereas the Hamiltonian in Ref [27] is frustrated. Another difference is that we associate two qubits per circuit location that are projected onto an EPR state, cf. Figure 3.

In Ref. [28] Bartlett and Rudolph show using PEPS that a fault-tolerant cluster state, which is a universal resource state for MBQC, can be robustly encoded into the ground state of a Hamiltonian consisting of planar, 2-local interaction terms. They also note that the approximation error can be interpreted as stochastic Pauli-noise and that the energy gap of their construction is independent of system size. The difference to our approach is that Bartlett and Rudolph use tensor networks to obtain a resource state that can be used for quantum computation via MBQC, whereas our scheme encodes a quantum computation into a tensor network.

In [29] Aharonov and Irani consider a mapping of classical computation into a CSP, which we may think of as a classical local Hamiltonian. More concretely, they consider a two-dimensional $L \times L$ grid with translation invariant constraints and show that approximating the ground state energy to an additive $\Theta(\sqrt[4]{L})$ is NEXP-complete. They do so by encoding a computation into a tiling problem. The computation is fault tolerant by running the same computation several times in parallel to enforce a large cost for an incorrect computation. In contrast, our model is fully quantum and thus requires the quantum fault tolerance theorem of Ref. [24].

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

Chapter

Background

Background

1) Hamiltonian Complexity

Definition 3.1. *The class* $\mathsf{QMA}_w[c, s]$ *is the class of promise problems* $A = (A_{\text{yes}}, A_{\text{no}})$ *with the property that, for every instance* x*, there exists a uniformly generated verifier quantum circuit* V_x *with the following properties:* V_x *is of size* $\text{poly}(|x|)$ *and acts on an input state* $|0^{\otimes m}\rangle$ *together with a witness state* $|\xi\rangle$ *of size* w *supplied by an all-powerful prover, with both* $m, w = \text{poly}(|x|)$*. Upon measuring the decision qubit* o*, the verifier accepts if* $o = 1$*, and rejects otherwise. If* $x \in A_{\text{yes}}$*, then* $\exists\, |\xi\rangle$ *such that* $\Pr[o = 1] \geq c$ *(completeness). If* $x \in A_{\text{no}}$*, then* $\forall\, |\xi\rangle$*,* $\Pr[o = 1] \leq s$ *(soundness), such that* $c - s \geq 1/\text{poly}(|x|)$*.*

It is well-known that the parameters c, s can be amplified, even without increasing the witness size.

Lemma 3.2 (Weak QMA amplification [1]). *For any* $r = \text{poly}(|x|)$, $\mathsf{QMA}_w[2/3, 1/3] = \mathsf{QMA}_{w'}[1 - 2^{-r}, 2^{-r}]$ *where* $w' = \text{poly}(w)$.

Lemma 3.3 (Strong QMA amplification [30]). *For any* $r = \text{poly}(|x|)$, $\mathsf{QMA}_w[2/3, 1/3] = \mathsf{QMA}_w[1-2^{-r}, 2^{-r}]$.

Background

1) Hamiltonian Complexity

Definition 3.4 (k-Local Hamiltonian problem). ***Input:*** *$H_1, H_2, \ldots, H_T$ set of $T = \text{poly}(n)$ Hermitian matrices with bounded spectral norm $\|H_i\| \leq 1$ acting on the Hilbert space of n qubits. In addition, each term acts nontrivially on at most k qubits and is described by $\text{poly}(n)$ bits. Furthermore, we are given two real numbers a, b (described by $\text{poly}(n)$ bits) such that $b - a > 1/\text{poly}(n)$.* ***Output:*** *Promised either the smallest eigenvalue of $H = H_1 + H_2 + \ldots H_T$ is smaller than a or all eigenvalues are larger than b, decide which case it is. We denote this problem by k-$\mathsf{LH}[a, b]$, or sometimes, k-$\mathsf{LH}(b - a)$.*

The k-LH is in QMA for any $k = O(\log n)$ (see e.g., Theorem 1 in [13]). Furthermore, Kitaev showed in his seminal work [1] that 5-LH is QMA-complete.

Theorem 3.5 (Kitaev [1]). *Any $\mathsf{QMA}_w[c, s]$ protocol involving an n-qubit verifier circuit with $T = \text{poly}(n)$ gates can be turned into a 5-$\mathsf{LH}[a, b]$ on $\text{poly}(n)$ qubits with $a = O((1 - c)/T)$ and $b = \Omega((1 - \sqrt{s})/T^3)$.*

We will often simply write QMA, LH when the parameters are unimportant or clear from context.

Next, we need the following lemmas in this work.

Lemma 3.6 (Detectability lemma [31]). *Let $\{Q_1, \ldots, Q_m\}$ be a set of projectors and $H = \sum_{i=1}^m Q_i$. Assume that each Q_i commutes with all but g others. Given a state $|\psi\rangle$, define $|\phi\rangle := \prod_{i=1}^m (\mathrm{I} - Q_i) |\psi\rangle$, where the product is taken in any order, and let $e_\phi = \langle\phi| H |\phi\rangle / \|\phi\|^2$. Then*

$$\|\phi\|^2 \leq \frac{1}{e_\phi / g^2 + 1}.$$

Background

1) Hamiltonian Complexity

Lemma 3.7 (Quantum union bound [32]). *Consider the same setting as in Lemma 3.6, but this time we do* not *require each Q_i to commute with at most g others. It holds that*

$$\|\phi\|^2 \geq 1 - 4 \langle\psi| H |\psi\rangle .$$

Lemma 3.8 (Jordan's lemma [33]). *Given two projectors Π_1, Π_2 acting on a d-dimensional complex vector space $\mathcal{H}$, there exists a change of basis such that $\mathcal{H}$ is decomposed as a direct sum of one- or two-dimensional mutually orthogonal subspaces $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, such that both the projectors leave the subspaces invariant. In other words, we can write $\Pi_1 = \sum_i a_i |u_i\rangle\langle u_i|$ and $\Pi_2 = \sum_i b_i |v_i\rangle\langle v_i|$, with $|u_i\rangle, |v_i\rangle \in \mathcal{H}_i$ and $a_i, b_i \in \{0,1\}$.*

Lemma 3.9 (Geometric lemma [1]). *Let A, B be nonnegative Hermitian operators and* g.s.(A), g.s.(B) *be their null subspaces such that the angle between them is $\theta > 0$. Suppose further that no nonzero eigenvalue of A or B is smaller than γ. Then $A + B \geq \gamma(1 - \cos\theta)$.*

Background

2) Fault tolerance

CSS codes The most studied class of quantum codes are Calderbank-Shor-Steane (CSS) codes, which are specified by an $r_X \times n$ matrix H_X, whose rows represent X-checks and a $k \times n$ matrix L_X whose rows represent Pauli X-logicals. The Z checks are $r_Z \times n$ matrix $H_Z = \ker \begin{pmatrix} H_X \\ L_X \end{pmatrix}$ and Z-logicals are $k \times n$ matrix L_Z. The codewords in the logical Z basis are

$$|v\rangle_L = \sum_{u \in \mathbb{F}_2^{r_X}} |uH_X + vL_X\rangle \qquad \text{(strings of 0 and 1)} \tag{9}$$

For CSS codes, qubit-wise CNOTs between two code blocks apply logical CNOTs between corresponding pairs of logical qubits. Indeed,

$$\begin{aligned} |v\rangle_L |w\rangle_L &= \sum_{u \in \mathbb{F}_2^{r_X}} |uH_X + vL_X\rangle \sum_{u' \in \mathbb{F}_2^{r_X}} |u'H_X + wL_X\rangle \\ &\longrightarrow \sum_{u \in \mathbb{F}_2^{r_X}} |uH_X + vL_X\rangle \sum_{u' \in \mathbb{F}_2^{r_X}} |u'H_X + (v+w)L_X\rangle \\ &= |v\rangle_L |v+w\rangle_L . \end{aligned}$$

We note that the existence of quantum codes does not guarantee that quantum computing can be made robust against noise. Manipulating the encoded states via an error prone process leads to errors spreading and it is this spread of errors that needs to be controlled.

Background

2) Fault tolerance

Theorem 3.10 ([24], Theorem 12). *There exists a noise threshold $\eta_c > 0$ such that for any $\eta < \eta_c$, $\varepsilon > 0$ the following holds. For any n-qubit quantum circuit C with s gates, ℓ locations, and depth D, there exists a quantum circuit $\tilde{C}$ of size $s\,\mathrm{polylog}(\ell/\varepsilon)$ (no measurements or classical operations are required) and depth $D\,\mathrm{polylog}(\ell/\varepsilon)$ operating on $n\,\mathrm{polylog}(\ell/\varepsilon)$ qubits such that in the presence of local depolarizing noise with error rate $\eta < \eta_c$, the encoded output of $\tilde{C}$ is ε-close to that of C.*

The theorem above does assume all-to-all connectivity, i.e. gates can be applied on arbitrary sets of qubits. We can also constrain the circuit to only operate locally on a d-dimensional grid of qubits, so that two qubit gates are only applied between neighbours on the grid. Note that an arbitrary circuit can be turned into a d-dimensional circuit by introducing SWAP gates and ancilla qubits, leading to the following result for any $d \geq 1$.

Corollary 3.11 ([24], Theorem 13). *There exists a noise threshold $\eta_c > 0$ such that for any $\eta < \eta_c$, $\varepsilon > 0$, and $d \geq 1$ the following holds. For any d-dimensional n-qubit quantum circuit C with s gates, ℓ locations, and depth D, there exists a d-dimensional quantum circuit $\tilde{C}$ of size $s\,\mathrm{polylog}(\ell/\varepsilon)$ (no measurements or classical operations are required) and depth $D\,\mathrm{polylog}(\ell/\varepsilon)$ operating on $n\,\mathrm{polylog}(\ell/\varepsilon)$ qubits such that in the presence of local depolarizing noise with error rate $\eta < \eta_c$, the encoded output of $\tilde{C}$ is ε-close to that of C.*

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
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Chapter III

Connection to quantum PCP conjecture

03. Connection to Quantum PCP conjecture

1) Adversarially noisy computation

noise → i.i.d. noise → g.s.
noise → adversarial noise → not g.s.

Recall that we consider a circuit W with initial state of the form $|0^a\rangle|\xi\rangle$, where $|\xi\rangle$ is any $(n-a)$-qubit representing the witness coming from the QMA prover (for BQP computations, $|\xi\rangle$ would be empty). Our starting point is the intuition that violated terms in H_{parent} should correspond to faults in the circuit. Informally, violated terms in H_{in} should correspond to errors at a set of locations, denoted S_0, in the qubit initialization step, violated terms in the first layer of H_{prop} should correspond to gate errors at locations, denoted S_1, in the circuit's first layer, and so on. The rest of this section is to make this connection between violated Hamiltonian terms and gate faults rigorous. However, these gate faults are adversarial in the sense that the faulty locations are chosen arbitrarily by the adversary. Thus, let us define the notion of adversarially noisy computations.

1. low energy state(not g.s.) → There is a adversarial noise.
2. difficulty to answer → adversarial fault tolerance

03. Connection to Quantum PCP conjecture

1) Adversarially noisy computation

noise → i.i.d. noise → g.s.

noise → adversarial noise → not g.s.

Definition 4.1. *Suppose* $S = \{S_0, \ldots, S_D\}$*, where* $S_\ell \subseteq [n]$ *for* $0 \leq \ell \leq D$*, is a set of locations in a depth-*D n*-qubit circuit. We define* $\mathrm{err}(S) = \{\vec{E} \in \mathcal{P}^{\otimes n(D+1)} : \mathrm{loc}(\tilde{E}_\ell) \subseteq S_\ell, 0 \leq \ell \leq D\}$ *to be the set of Pauli errors supported within the set of locations* S*.*

Definition 4.2 (Adversarial computations)**.** *For any sets of locations* $S_\ell \subseteq [n]$*, for* $0 \leq \ell \leq D$ *in the circuit* W*, a state* $|\psi\rangle$ *is said to be an adversarial computation at locations* $S = \{S_0, S_1, \ldots, S_D\}$ *if*

$$|\psi\rangle \in \mathrm{adv}(W, S) \triangleq \mathrm{span}\{\tilde{E}_D W_D \ldots \tilde{E}_1 W_1 \tilde{E}_0 |0^a\rangle |\xi\rangle : \forall |\xi\rangle, \vec{E} \in \mathrm{err}(S)\}.$$

We consider adversarial computations such that at most εn *adversarial errors are present in the circuit. In particular, we say a state* $|\psi\rangle$ *is an* ε*-noisy state if*

$$|\psi\rangle \in \mathrm{adv}_\varepsilon(W) \triangleq \mathrm{span}\{\mathrm{adv}(W, S) : \sum_{\ell=0}^{D} |S_\ell| \leq \varepsilon n\}.$$

A mixed state ρ *is* ε*-noisy if it is a convex combination of* ε*-noisy pure states.*

03. Connection to Quantum PCP conjecture

1) Adversarially noisy computation

noise → i.i.d. noise → g.s.

noise → adversarial noise → not g.s.

Theorem 4.3 (Soundness). *Suppose the depth $D = O(\log n)$ and consider any injectivity parameter $\delta = O(D^{-0.51})$. For any state $|\psi\rangle$ with energy density $\frac{\delta^{200D}}{D+1}$ with respect to H_{parent}, the reduced ψ_{out} in the output column is $\frac{1}{10}$-close in trace distance to a $400\delta^2 D$-noisy mixed state.*

Theorem 4.4 (Combinatorial soundness). *There exists a constant ε_0 such that the following holds. Consider any injectivity parameter $\delta = O(D^{-0.51})$ and any $10\delta\sqrt{D} < \varepsilon < \varepsilon_0$. Then for any state $|\psi\rangle$ that satisfies all but $\frac{\varepsilon}{D+1}$ fraction of terms in H_{parent}, the reduced state ψ_{out} in the output column is e^{-99n}-close in infidelity to an 8ε-noisy mixed state.*

Remark 4.5. *The theorem statements and proofs below are presented assuming all n qubits are intialized at the beginning of the computation for simplicity. However, they can be readily adapted to the setting where qubits are initialized at varying times such as in quantum fault tolerance. In this case, D is defined to be the longest elapse time between an output qubit and the initialization of any qubit causally connected to it.*

03. Connection to Quantum PCP conjecture

2) Combinatorial soundness

noise → i.i.d. noise → g.s.

noise → adversarial noise → not g.s.

4.2 Proof of Theorem 4.4 (Combinatorial soundness)

Proof idea: The combinatorial state $|\psi\rangle$ has the property that the (unnormalized) state $\Lambda^{\otimes nD}|\psi\rangle$ has a nice form - $\left(\bigotimes_{i\notin S_0}|0\rangle_i\right)\left(\bigotimes_{\mathrm{loc}(U)\notin S}|\Phi_U\rangle\right)\otimes|\psi''\rangle$. This means that we have the correct state $|0\rangle$ or $|\Phi_U\rangle$ corresponding to the satisfied Hamiltonian terms and an arbitrary state $|\psi''\rangle$ at the violated terms. If $|\psi''\rangle$ were of the form $\bigotimes_{j\in S}|\Phi_{U_j}\rangle$ for some 2-qubit unitaries U_j, then we could simply view the state $|\psi\rangle$ as encoding the circuit with iid noise on non-faulty locations, and adversarial noise at faulty locations. This would be a perfectly fine combination of stochastic error and small number of adversarial errors. But $|\psi''\rangle$ can be a superposition of the states of above form, which can arbitrarily correlate the noise at non-faulty locations! We appeal to the injectivity of the local maps Λ to argue that despite this possible correlation of noise at non-faulty locations, the fraction of errors stays at $O(\delta^2)$ (with high probability). Thus a damaging situation, for example all the non-faulty locations experiencing a Pauli error, continues to occurs with very small probability.

03. Connection to Quantum PCP conjecture

3) Soundness

noise → i.i.d. noise → g.s.

noise → adversarial noise → not g.s.

The previous subsections characterize states with sufficiently low energy as encoding a noisy execution of the quantum circuit. We can also consider other basic properties of the parent Hamiltonian such as spectral gap. Here we give a lower bound on the spectral gap, which will be used to give a new proof of QMA-completeness of the local Hamiltonian problem in later sections.

Theorem 4.16. *Suppose that all gates in H_{prop} and input check terms in H_{in} have locality k. Then the spectral gap of H_{parent} is lower bounded by $\gamma = \delta^{8k(D+1)}/\operatorname{poly}(nD)$. More generally, if we vary the injectity parameter δ and gate locality in the PEPS, then the factor δ^{8kD} is replaced by the product of the injectivity parameters across the depth of the circuit, i.e., $\gamma = \frac{1}{\operatorname{poly}(nD)} \prod_{\ell=0}^{D} \delta_\ell^{8k_\ell}$, where δ_ℓ and k_ℓ are the injectivity and gate locality in layer ℓ ($\ell = 0$ corresponds to initializations).*

Proof idea: We tak[...] suitable rotated basis. $n(2D+1)$-qubit unita[...] of $H_{\mathrm{parent}} = H_{\mathrm{in}} + H_{\mathrm{p}}$[...] states (Claim 2.1) are

Our starting observation is based on a surprising effect - despite the fact that H_{in} enforces $|0\rangle^{\otimes n}$ on the first column, the state $|0\rangle^{\otimes n}$ appears on the last column in $V^\dagger |\Psi_{W,\xi}\rangle$. We view this as a *teleportation of* H_{in}, highlighting that its a noiseless teleportation under 'zero energy' constraint, despite the tensor network performing noisy gate-by-gate teleportation. Given this, we focus on establishing two properties for low energy states:

- *Robust teleportation of H_{in}:* Upon rotating with V, the low energy states should look like $|0\rangle$ in most of the qubits (that do not include witness qubits) in the last column. This amounts to H_{in} effectively acting on the last column under the constraint of low energy.
- The number of Pauli errors is small enough in a low energy state.

03. Connection to Quantum PCP conjecture

4) Spectral gap lowerbound

noise → i.i.d. noise → g.s.
noise → adversarial noise → not g.s.

The previous subsections characterize states with sufficiently low energy as encoding a noisy execution of the quantum circuit. We can also consider other basic properties of the parent Hamiltonian such as spectral gap. Here we give a lower bound on the spectral gap, which will be used to give a new proof of QMA-completeness of the local Hamiltonian problem in later sections.

Theorem 4.16. *Suppose that all gates in H_{prop} and input check terms in H_{in} have locality k. Then the spectral gap of H_{parent} is lower bounded by $\gamma = \delta^{8k(D+1)} / \operatorname{poly}(nD)$. More generally, if we vary the injectity parameter δ and gate locality in the PEPS, then the factor δ^{8kD} is replaced by the product of the injectivity parameters across the depth of the circuit, i.e., $\gamma = \frac{1}{\operatorname{poly}(nD)} \prod_{\ell=0}^{D} \delta_\ell^{8k_\ell}$, where δ_ℓ and k_ℓ are the injectivity and gate locality in layer ℓ ($\ell = 0$ corresponds to initializations).*

03. Connection to Quantum PCP conjecture

5) Adversarial fault tolerance against inverse-polynomial adversarial noise

Here we note that a repetition argument similar to [36] (see Appendix B) also suffices to protect a computation against an inverse-polynomial fraction of adversarial noise for any desired polynomial, at the cost of increasing the circuit size by a corresponding polynomial factor.

Classical case:

Given a circuit C on n bits with T gates, lets us run the circuit in parallel k times, for k to be chosen shortly. Let $C_1, C_2, \ldots C_k$ be these runs of C. The repeated circuit has kT gates. For a $\delta > 0$, we would like to protect against $(kT)^{1-\delta}$ adversarial errors. Note that even if there was at most 1 error per C_i, the number of circuits with no error is $k - (kT)^{1-\delta} = k\left(1 - \frac{(T)^{1-\delta}}{k^\delta}\right)$. Choosing $k = (100T)^{\frac{1}{\delta}}$, we see that at least $0.99k$ circuits have no error and the output of the computation can be read by considering the majority value.

In fact, we do not need to do fault tolerant majority computation. We simply put a Hamiltonian $H_{out} = \sum_i |1\rangle\langle 1|_i$ on the k output bits. This Hamiltonian penalizes if most of the outputs are 1. Further, note that for any constant δ, this is a polynomial sized transformation.

03. Connection to Quantum PCP conjecture

5) Adversarial fault tolerance against inverse-polynomial adversarial noise

Quantum case:

Identical argument works in the quantum case if the adversarial error does not occur in superposition and the quantum circuit computes the correct outcome with probability 0.9. This happens in the case of combinatorial soundness, where the error locations are fixed. It is far from clear if general superposition over low weight errors can be handled. But at the same time, the low energy states may not admit an arbitrary superposition over errors. We leave this understanding for the future work.

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

Chapter IV

New proof of QMA-completeness of local Hamiltonian

04. New proof of QMA-completeness of local Hamiltonian

1) Verifying QMA via shallow circuits

As shown in Section 4, the parent Hamiltonian robustness properties only depend on circuit depth, so it is desirable to restrict our attention to shallow circuits. Here we show that any QMA protocol can be replaced by one involving a constant depth quantum circuit followed a logarithmic depth classical circuit. The high-level idea is to first use the Feynman-Kitaev mapping to turn an arbitrary QMA protocol into a local Hamiltonian, and then construct a short-depth QMA circuit to measure the energy of the resulting Hamiltonian. For this, we need a low-degree version of the FK mapping.

Claim 5.1 (Degree reduction for FK Hamiltonian). *Any* QMA *protocol involving an n-qubit verifier circuit V with $T = \mathrm{poly}(n)$ two-qubit gates can be mapped into a* 5-LH$[a, b]$ *on* $\mathrm{poly}(n)$ *qubits with* $a = 2^{-\mathrm{poly}(n)}$ *and* $b = a + 1/\mathrm{poly}(n)$. *Furthermore, each qubit is involved in at most 7 terms in the Hamiltonian.*

Claim 5.3 (Log-depth QMA). *Any* QMA *protocol involving an n-qubit verifier circuit V with $T = \mathrm{poly}(n)$ two-qubit gates can be converted into a $O(\log n)$-depth* QMA *protocol on* $\mathrm{poly}(n)$ *qubits, whose completeness is $1 - 2^{-r}$ and soundness is 2^{-r} with $r = \mathrm{poly}(n)$. More specifically, the $O(\log n)$-depth circuit involves a constant-depth quantum circuit that ends with computational basis measurements, followed by a $O(\log n)$-depth classical circuit.*

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

$$H_{\text{stab}} = \sum_{\text{stabilizer } j} \Lambda^{\otimes N(j)} \frac{1 - S_j}{2} \Lambda^{\otimes N(j)}$$

$$H_{\text{total}} = \underbrace{H_{\text{in}} + H_{\text{prop}} + (\text{optionally } H_{\text{stab}})}_{H_{\text{parent}}} + C H_{\text{out}}$$

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

Claim 6.1 (QMA-hardness from a fault-tolerant verifier). *Consider a (noiseless)* QMA *verifier circuit W where p commuting binary projective measurements are performed at the end, such that either (completeness) there exists a witness input state such that all of the p measurements return 1 with probability at least c, or (soundness) for any witness state, with probability at least $1-s$, one of the measurements returns 0. Suppose there exists a fault-tolerant version W_{FT} such that its PEPS parent Hamiltonian (H_{parent} in Equation (68)) has spectral gap at least γ and the probability of a logical error in the PEPS ground state is at most δ_L, such that $\max\{1-c, \delta_L\} < \frac{1-s}{16p^2}$. Let $H_{\mathrm{total}} = H_{\mathrm{parent}} + \gamma H_{\mathrm{out}}$, where H_{out} has p terms corresponding to the p output measurements in W_{FT}. Then, the following holds:*

- *In the completeness case, H_{total} has an eigenvalue smaller than $\frac{\gamma(1-s)}{8p}$.*
- *In the soundness case, all eigenvalues of H_{total} are at least $\frac{\gamma(1-s)}{4p}$.*

In other words, determining the ground energy of H_{total} to precision $\frac{\gamma(1-s)}{8p}$ is QMA*-hard.*

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

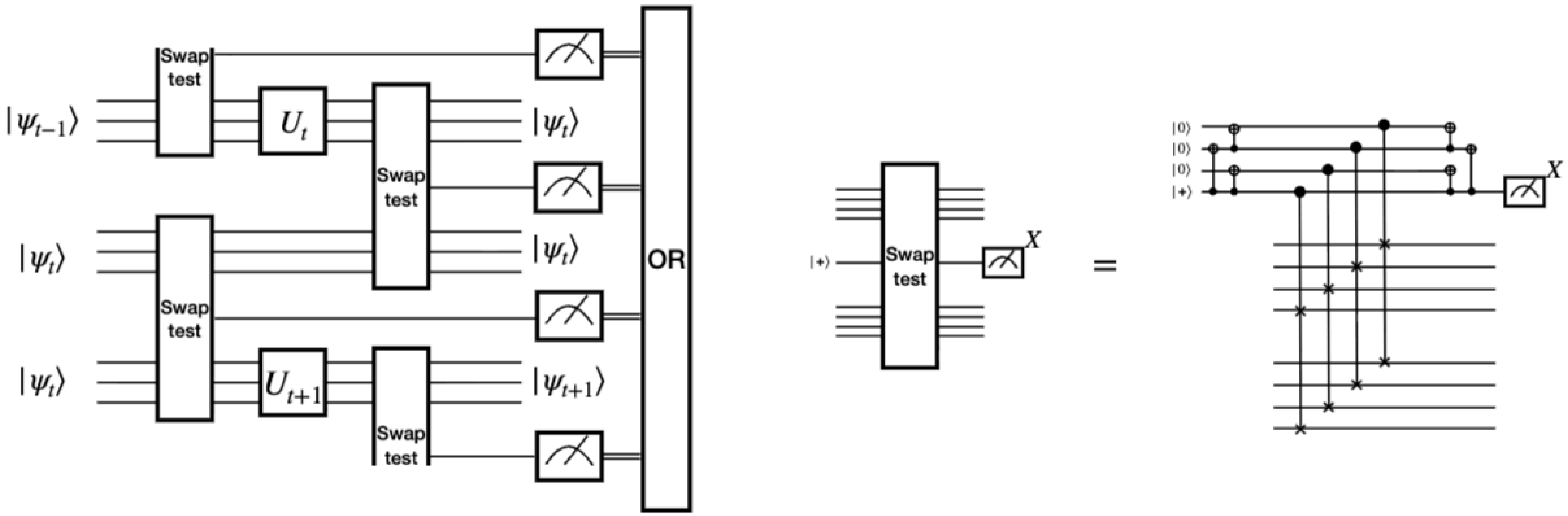

Figure 8: *Left*: Parallelized QMA protocol with SWAP tests in Claim 6.2. The verifier expects all measurements to output 1. *Right*: Log-depth implementation of SWAP test.

$$H_{\text{total}} = H_{\text{parent}} + \gamma H_{\text{out}} \geq \frac{1-s}{4p}\gamma$$

We note that the condition $1-c < \frac{1-s}{16p^2}$ is a simple technical requirement that can be satisfied by preprocessing the original (noiseless) circuit W by QMA amplification, while the condition $\delta_L < \frac{1-s}{16p^2}$ just means that the noise needs to be sufficiently suppressed. Finally, the reason for the energy scaling factor γ in front of H_{out} is because even in the completeness case, the logical error rate δ_L could create a large penalty on H_{out}, so we set the energy scale of γ to bring this penalty below the spectral gap. □

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

Theorem 6.5. *The problem of deciding whether the ground energy density of $O(\log n)$-local Hamiltonians is $\leq a$ or $\geq a + 1/\operatorname{poly}(n)$, for some given number a, is* QMA*-complete.*

Proof. As noted in Section 3, the log-local Hamiltonian problem is in QMA. We here show it is QMA-hard. Let n and $D = O(\log n)$ be the width and depth of the circuit W from Claim 6.2. Our goal is to construct a fault-tolerant version of the circuit W in Figure 8 while keeping the depth $O(\log n)$, and then apply Claim 6.1. As discussed in the previous subsection, we can offload the measurements at the end of the circuit to the output Hamiltonian terms in H_{out}, so we don't need to make that part fault-tolerant.

For each qubit in W, we simulate it by an instance of the linear-distance CSS QECC family in Lemma 6.4 of block size $m = O(\log nD) = O(\log n)$. Without loss of generality, we assume the gates $U_1, \ldots U_T$ in W are CCZ, Hadamard, and CNOT. The SWAP tests can also be *exactly* implemented with this gate set. We will use the following gadgets to fault-tolerantly simulate the circuit W in Figure 8 using a circuit W_{FT} of width $n_{\text{FT}} = O(n \log n)$ and depth $D_{\text{FT}} = O(D) = O(\log n)$:

1. (Offline) Logical state preparations of $|0\rangle_L$, $|+\rangle_L$, $\text{CZ}_L\,|{+}{+}\rangle_L$, $\text{CCZ}_L\,|{+}{+}{+}\rangle_L$.
2. Error correction gadget that incurs constant space and time overheads per round.
3. Logical CZ, CCZ, and Hadamard gates via gate teleportation.
4. Transversal logical CNOT gate.

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

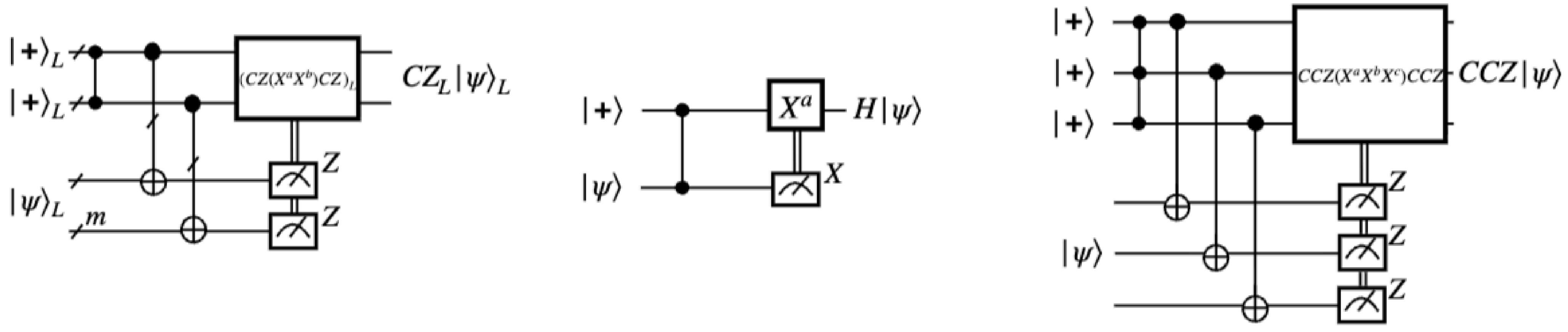

Figure 9: *Left*: CZ gate teleportation using the ancilla state $CZ|{+}{+}\rangle$. The correction $CZ(X^aX^b)CZ$ is a Pauli operation. *Middle*: Hadamard gate teleportation. *Right*: CCZ gate teleportation. The correction $CCZ(X^aX^bX^c)CCZ$ is a product of Pauli operators and CZ. For example, $CCZ(X\otimes I\otimes I)CCZ = X\otimes CZ$. The logical X/Z measurement can be transversally done in the physical X/Z basis for CSS codes.

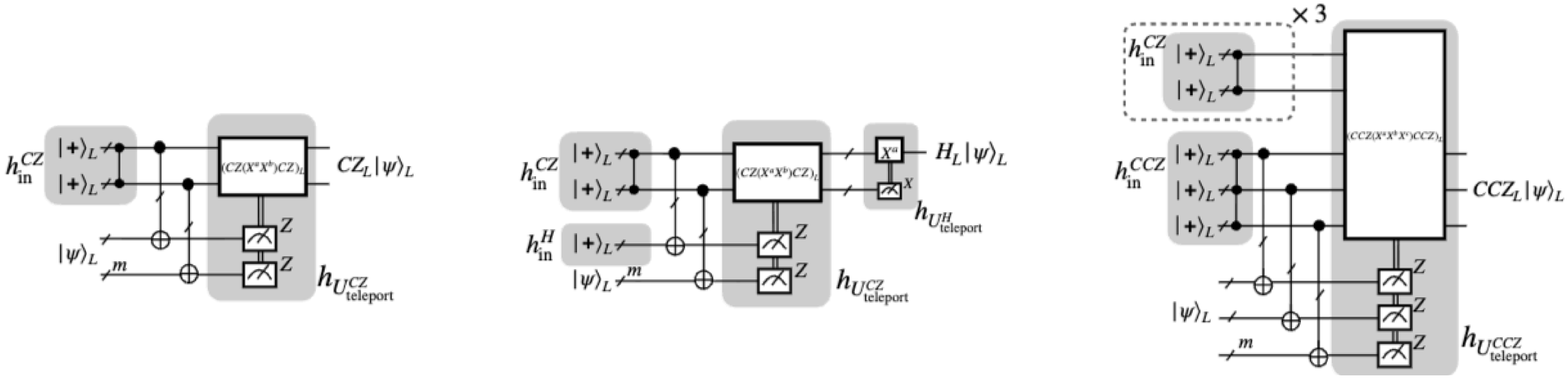

Figure 10: Fault-tolerant realizations of the encoded versions of the teleportation circuits in Figure 9 using $O(m)$-local gates and Hamiltonian terms. The Hadamard circuit uses CZ as a subroutine. The CCZ circuit adaptively uses up to 3 CZ's within its Clifford correction.

04. New proof of QMA-completeness of local Hamiltonian

2) Proof of QMA-hardness of local Hamiltonian problem

Claim 6.6. *The parent Hamiltonian corresponding to W_{FT} has spectral gap at least $\gamma = 1/\operatorname{poly}(n)$.*

Proof. We apply the version of Theorem 4.16 that allows gate locality to vary in the circuit. Theorem 4.16 states that the spectral gap is lowerbounded by $\gamma = \frac{1}{\operatorname{poly}(n_{\mathrm{FT}} D_{\mathrm{FT}})} \prod_{\ell=0}^{D_{\mathrm{FT}}} \delta^{8k_\ell}$, where k_ℓ is the gate locality at layer ℓ of W_{FT}. But W_{FT} has depth $D_{\mathrm{FT}} = O(\log n)$ and only a constant number of layers have $O(\log n)$ locality. Thus, $\gamma = 1/\operatorname{poly}(n)$. □

For the case of $O(1)$-local Hamiltonian, we can also prove QMA-hardness with an inverse superpolynomial promise gap.

Theorem 6.7. *The problem of deciding whether the ground energy density of $O(1)$-local Hamiltonians is $\leq a$ or $\geq a + n^{-O(\log\log n)}$, for some given number a, is* QMA*-hard.*

- Connection to quantum PCP conjecture
- New proof of QMA-completeness of local Hamiltonian
- Complexity of injective tensor networks

Chapter V

Complexity of injective tensor networks

05. Complexity of injective tensor networks

1) Computational complexity of injective PEPS

Definition 7.1 (PEPS). *A* projected entangled pair state (PEPS) *is any (unnormalized) state that can be obtained by the following procedure: consider a graph and associate to each vertex v as many D-dimensional spins as there are edges incident to v. Assume that the spins associated to the end points of an edge form maximally entangled states $|\mathrm{EPR}_D\rangle = \sum_{i=1}^{D} |i\rangle |i\rangle$. The PEPS is obtained by applying a linear map $P_v : \mathbb{C}^D \otimes \cdots \otimes \mathbb{C}^D \to \mathbb{C}^d$ at each vertex v. Without affecting the computational complexity, we further allow the virtual states to be any maximally entangled states of the form $(\mathrm{I} \otimes U) |\mathrm{EPR}_D\rangle$. We can also assume $\|P_v\| \leq 1$.*

Definition 7.2 (Injective PEPS). *A PEPS on n spins is $\delta(n)$-*injective *if the local maps P_v are non-singular matrices with singular values bounded from below by $\Omega(\delta(n))$.*

Theorem 7.3. *Preparing constant-injective PEPS states in two or higher dimensions with bond dimension $D \geq 4$ and physical dimension $d \geq 4$ allows solving* BQP*-hard problems.*

Theorem 7.5. *For constant-injective PEPS states in two or higher dimensions with bond dimension $D \geq 4$ and physical dimension $d \geq 4$, evaluating local observable expectation values to $O(1)$ additive error is* BQP*-hard.*

05. Complexity of injective tensor networks

1) Computational complexity of injective PEPS

Task	PEPS	Injective PEPS
State preparation	PostBQP-complete	BQP-hard
Multiplicative-error contraction	#P-complete	#P-complete*
Additive-error contraction	BQP-hard	BQP-hard

Table 2: Computational complexity of general PEPS [15] and constant-injective PEPS. *The #P-hardness of injective PEPS requires a specific non-local observable in Theorem 7.6.

Theorem 7.6. *For constant-injective PEPS states in two or higher dimensions with bond dimension $D \geq 4$ and physical dimension $d \geq 4$, evaluating the expectation value of a tree tensor network observable to $O(1)$-multiplicative error is #P-hard.*

Open questions

Open questions

Can we achieve a soundness of 1/poly(D)?

Pauli error

$$h_A^{\text{low}} = \sum_{\vec{P} \in \mathcal{P}^A : |\vec{P}| \geq 10\delta^2 \cdot \frac{D}{\delta^4}} |\mathbf{\Phi}_{\vec{P}}\rangle\langle\mathbf{\Phi}_{\vec{P}}|$$

Total fraction of Pauli error

$$\frac{1}{nD}\sum_j (I - \Phi_{I_j}) \preceq O(1)\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\delta^4}{D}\sum_{j\in A_i}(I - \Phi_{I_j})\right) \preceq O(1)\cdot 10\delta^2 I + \frac{O(1)}{m}\sum_{i=1}^{m} h_{A_i}^{\text{low}}$$

Open questions

- In the introduction and Appendix A, we outline a connection between polylog-PCP and adversarial fault tolerance in the classical setting. It is expected that adversarial fault tolerance may use good classical codes, but we do not see a clear use of local decodability. Could classical polylog-PCP be achieved without strong reliance on local decodability?

- Can the depth of BQP circuits be reduced to polylogarithmic in the input size? This does not follow from the depth reduction of QMA due to the presence of witness. Thus, the heart of the question is if the ground state of the tensor network Hamiltonian can be prepared in low depth when witness is absent. One possibility is to run an adiabatic algorithm tuning δ from 1 to a smaller value. The spectral gap in this process is likely small – we can show a spectral gap lower bound of $\Omega(e^{O(D)}/\text{poly}(nD))$ in Theorem 4.16. But suppose that we go ahead and tune δ adiabatically for small duration, can we argue that we end up in a low energy state of the parent Hamiltonian (not necessarily the ground state as in standard adiabatic computation)? If that is the case, we would still encode the answer to the computation if we started from a fault-tolerant circuit.

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Thank you for listening

조정빈