Models of quantum complexity growth

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Motivation & Overview

- When we say a quantum state is highly complex, we mean there is no easy way to prepare the state, but how can we be sure?
- By enumerating all the quantum circuits that approximate a specified state…

  → difficult to obtain a useful lower bound

- Goal is to strengthen the evidence supporting the cojective below

Conjecture 1 (by Brown and Susskind)

Most local random circuits of size T have a complexity that scales linearly in T for an exponentially long time

Motivation & Overview

When we say a quantum state is highly complex, we mean there is no easy way to prepare the state, but how can we be sure? Perhaps we were not clever enough to think of an ingenious short-cut that prepares the state efficiently. It's not possible in practice to enumerate all the quantum circuits that approximate a specified state to find one of minimal size. For that reason, it is quite difficult to obtain a useful lower bound on the complexity of the quantum state prepared by a specified many-body Hamiltonian in a specified time. It is reasonable to expect that, for a chaotic Hamiltonian H and an unentangled initial state, the complexity grows linearly in time for an exponentially long time, but we do not have the tools to prove it from first principles for any particular H.

One possible approach is to rely on highly plausible complexity theory assumptions to derive nontrivial conclusions about the complexity of states generated by particular circuits or Hamiltonians [8–10]. Another is to consider ensembles of circuits, and to derive lower bounds on complexity which hold with high probability when samples are selected from these ensembles. We follow the latter approach here, drawing inspiration from recent work

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How This Paper Strengthens Conjecture 1?

- Random circuits that generate approximate unitary k-designs necessarily contain many unitaries with strong complexity $\Omega(k)$.
- The distribution over design elements cannot concentrate on a few low-complexity unitaries (anti-spikiness).
- Therefore, as the design order k grows with time, the number and typical complexity of unitaries grow accordingly.

→ This provides rigorous evidence for Conjecture 1.

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- Therefore, as the design order k grows with time, the number and typical complexity of unitaries grow accordingly.

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Theorem 1 (informal statement)

Let $\{p_i, U_i\}$ be an approximate unitary k-design. Then, a randomly selected (according to the weights) element is very likely to have strong circuit complexity $\approx k$

Lemma 1 (anti-spikiness)

Let $\{p_i, U_i\}$ be an approximate unitary k-design. Then, the associated weight distribution cannot be too spiky: $\max_i p_i \lesssim k!\, d^{-2k}$.

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Corollary 1

Any approximate k-design contains exponentially many (in k) unitaries that have circuit complexity $\Omega(k)$.

N. Hunter Jones shows that local circuits of size $T = O(n^2 k)$ form approximate k-designs in the limit of large local dimension (Hilbert space dimension $d = q^n$ with q large)

Corollary 2

The set of all local circuits of size T contains at least $\exp(\Omega(T))$ elements with strong complexity $\Omega(T)$, provided that the local dimension is sufficiently large: $q \geq q_0(T)$

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Lemma 5 (Restatement of Lemma 1). *Let $\mathcal{E} = \{p_i, U_i\}_{i=1}^N$ be an ϵ-approximate k design for $U(d)$. Then,*

$$\max_{1 \le j \le N} p_j \le (1+\epsilon)\frac{k!}{d^{2k}} \quad \text{and} \quad N \ge \frac{d^{2k}}{(1+\epsilon)k!}. \tag{7.44}$$

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Untiary designs

Definition (Unitary k-design)

Let ν be a probability distribution defined over a set of unitaries $S \subseteq U(d)$. The distribution ν is unitary k-design if and only if:

$$\mathbb{E}_{V\sim\nu}\left[V^{\otimes k} O V^{\dagger\otimes k}\right] = \mathbb{E}_{U\sim\mu_H}\left[U^{\otimes k} O U^{\dagger\otimes k}\right],$$

for all $O \in \mathcal{L}\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$.

Definition 4 (approximate k-design). *Fix $k \in \mathbb{N}$ and $\epsilon > 0$. A unitary ensemble $\mathcal{E} = \{p_i, U_i\}_{i=1}^N$ is an ϵ-approximate (unitary) k-design if the associated twirling channel $\mathcal{T}_{\mathcal{E}}^{(k)}(X) = \sum_{i=1}^n p_i U_i^{\otimes k} X (U_i^\dagger)^{\otimes k}$ obeys*

$$\left\| \mathcal{T}_{\mathcal{E}}^{(k)} - \mathcal{T}_{U}^{(k)} \right\|_\diamond \leq \frac{k!}{d^{2k}} \epsilon \,. \tag{7.43}$$

Here, $\mathcal{T}_U^{(k)}$ denotes the twirl over the full unitary group (7.32) (with respect to the Haar measure).

Strong state complexity

- Consider systems comprised of n qudits with local dimension q: $d = q^n$
- The maximally mixed state: $\rho_0 = \frac{\mathbb{I}}{d}$
- For any pure state $|\psi\rangle\langle\psi|$,

$$\frac{1}{2}\left\| |\psi\rangle\langle\psi| - \rho_0 \right\|_1 = 1 - \frac{1}{d}$$

- The optimal measurement is $M = |\psi\rangle\langle\psi|$ and does depend on the state in question

→ Such a measurement may be challenging to implement for states that we assign a high complexity to.

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- $\mathbb{H}_d$: the space of $d \times d$ Hermitian matrices
- $\mathrm{M}_r(d) \in \mathbb{H}_d$: the class of measurements that can be implemented by combining at most r 2-local gates from a fixed, universal gate set $\mathrm{G} \in U(4)$.

$$\begin{aligned} \beta_{qs}^{\#}(r, |\psi\rangle) = \text{maximize} \quad & |\mathrm{Tr}(M(|\psi\rangle\langle\psi| - \rho_0))| \\ \text{subject to} \quad & M \in \mathrm{M}_r(d) \end{aligned}$$

Strong state complexity

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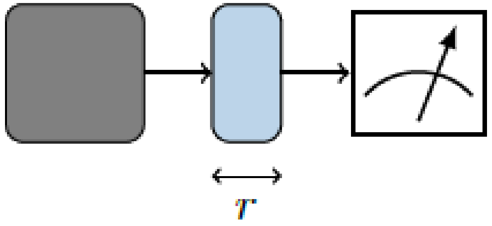
- $\beta^{\sharp}(r, |\psi\rangle) \longrightarrow \frac{1}{2}\| |\psi\rangle\langle\psi| - \rho_0 \|_1 = 1 - \frac{1}{d} \quad \text{as} \quad r \to \infty\,.$

Definition 2 (Strong state complexity)

Fix $r \in \mathbb{N}$ and $\delta \in (0,1)$. We say that a pure state $|\psi\rangle$ has strong δ-state complexity at most r if

$$\beta^{\#}_{qs}(r, |\psi\rangle) \geq 1 - \frac{1}{d} - \delta, \qquad \text{which we denote as } C_\delta(|\psi\rangle) \leq r.$$



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Lemma 2 (Strongness)

Suppose that $|\psi\rangle \in \mathbb{C}^d$ obeys $C_\delta(|\psi\rangle) \geq r + 1$ for some $\delta \in (0,1)$ and $r \in \mathbb{N}$. Then,

$$\min_{size(V) \leq r} \frac{1}{2} \left\| |\psi\rangle\langle\psi| - V|0\rangle\langle 0|V^\dagger \right\|_1 > \sqrt{\delta},$$

i.e. it is impossible to accurately produce $|\psi\rangle$ with fewer than r elementary gates.

Strong unitary complexity

- Define the complexity of unitary channels $\mathcal{U}(\rho) = U\rho U^\dagger$
- The completely depolarizing channel: $\mathcal{D}(\rho) = \rho_0 = \frac{\mathbb{I}}{d}$ for all states ρ.
- The diamond distance between $\mathcal{D}$ and any unitary channel is close to maximal:

$$\frac{1}{2}\|\mathcal{U} - \mathcal{D}\|_\diamond = 1 - \frac{1}{d^2}$$

- $G_{r'} \subset U(d^2)$: the set of all unitary circuits on $2n$ qudits (register+memory) that are comprised at most r' elementary gates.
- $M_{r''} \subset \mathbb{H}_d^{\otimes 2}$: the class of all two-outcome measurements on $2n$ qudits that require circuit size at most r'' to implement.

$$\begin{aligned} \beta_{\mathrm{qc}}^{\sharp}(r, U) = \text{maximize} \quad & \left|\mathrm{Tr}\left(M\left((\mathcal{U} \otimes \mathcal{I})\left(|\phi\rangle\langle\phi|\right) - (\mathcal{D} \otimes \mathcal{I})\left(|\phi\rangle\langle\phi|\right)\right)\right)\right| \\ \text{subject to} \quad & M \in \mathsf{M}_{r'}\,,\ |\phi\rangle = V|0\rangle\,,\ V \in \mathsf{G}_{r''}\,,\ r = r' + r'' \end{aligned}$$

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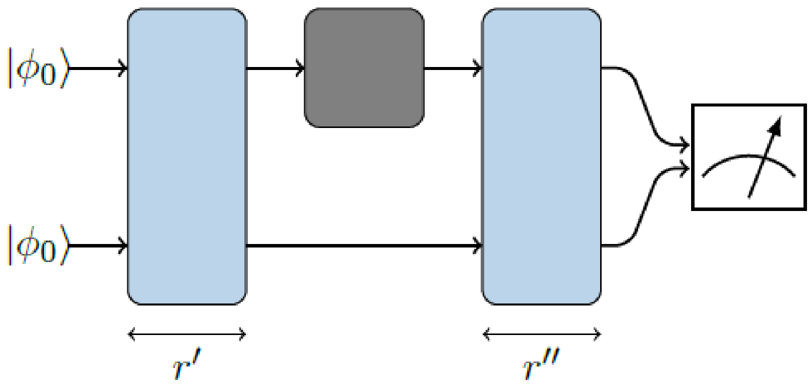
- $\beta^{\sharp}(r,U) \longrightarrow \frac{1}{2}\|\mathcal{U} - \mathcal{D}\|_{\diamond} = 1 - \frac{1}{d^2} \quad \text{as} \quad r \to +\infty$

Definition 3 (Strong unitary complexity)

Fix $r \in \mathbb{N}$ and $\delta \in (0,1)$. We say that a unitary $U \in U(d)$ has strong δ-unitary complexity at most r if

$$\beta^{\#}_{qc}(r,U) \geq 1 - \frac{1}{d^2} - \delta, \qquad \text{which we denote as } C_\delta(U) \leq r.$$



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Lemma 3 (Strongness)

Suppose that $U \in U(d)$ obeys $C_\delta(U) \geq r + 1$ for some $\delta \in (0,1)$, $r \in \mathbb{N}$ and measurement procedures that include the Bell-measurement $|\Omega\rangle\langle\Omega|$. Then,

$$\min_{size(V) \leq r} \frac{1}{2} \|\mathcal{U} - \mathcal{V}\|_\diamond > \sqrt{\delta},$$

i.e. it is impossible to accurately approximate U by circuits comprised of fewer than r elementary gates.

Complexity by design - State

Theorem 2 (State complexity growth)

Consider the set of (pure) states in $d = q^n$ dimensions that results from applying all unitaries associated with an ϵ-approximate $2k$-design to a fixed starting state $|\psi_0\rangle$.
Then, this set contains at least

$$\binom{d+k-1}{k}\left(\frac{1}{1+\epsilon} - 2dn^r|\mathrm{G}|^r\left(\frac{16k^2}{d(1-\delta)^2}\right)^k\right)$$

distinct states that obey $C_\delta(|\psi\rangle) \geq r+1$ each. Qualitatively, this number is of order $\left(\frac{d}{k}\right)^k$ as long as r obeys

$$r \lesssim \frac{k(n-2\log(k))}{\log(n)}$$

Complexity by design - Unitary

Theorem 3 (Unitary complexity growth)

A discrete approximate $2k$-design in $d = q^n$ dimension contains at least

$$\frac{d^{2k}}{k!}\left(\frac{1}{1+\epsilon} - 3d^2n^{2r}|\mathrm{G}|^r\left(\frac{1024k^4}{d(1-\delta)^2}\right)^k\right)$$

distinct unitaries that obey $C_\delta(U) \geq r+1$ each. Qualitatively, this number is of order $\left(\frac{d^2}{k}\right)^k$ as long as r obeys

$$r \lesssim \frac{k(n - 4\log(k))}{\log(n)}$$

Moment bounds

- To show most unitaries in a k-design are complex, we must bound how much measurement outcome can deviate from its average.
- Markov's inequality (for nonnegative RV S)

$$\Pr[S \geq \tau] = \Pr[S^k \geq \tau^k] \leq \mathbb{E}[S^k]/\tau^k$$

∴ The larger the moments we can control, the stronger this assertion becomes.

- For state complexity,

$$\mathbb{E}_{|\psi\rangle}\left[\left(\mathrm{Tr}(M|\psi\rangle\langle\psi|) - \mathbb{E}_{|\psi\rangle}\left[\mathrm{Tr}(M|\psi\rangle\langle\psi|)\right]\right)^k\right] \leq \left(\frac{k^2}{d}\right)^{k/2}$$

Theorem 4

Fix a bipartite input state $|\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and a measurement M of compatible dimension. Then,

$$\mathbb{E}_U\left[\left(\mathrm{Tr}\left((M(U \otimes I)|\phi\rangle\langle\phi|(U^\dagger \otimes I)\right) - \mathbb{E}_U[\mathrm{Tr}(M(U \otimes I)|\phi\rangle\langle\phi|(U^\dagger \otimes I))]\right)^k\right] \leq \frac{C_k (k!)^2}{d^{k/2}}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k} < \frac{4^k}{k}$ denotes the k-th Catalan number.

Use Theorem 2 and Theorem 3 to analyze concrete models

Local random circuits

- Focus on systems comprised of n qubits, i.e. $q = 2$ and $d = 2^n$
- $\mathrm{G} \subset U(4)$ be a finite universal gate set containing inverses
- Generate G-local random circuits by sequentially applying a random gate $g \in \mathrm{G}$ to a randomly selected pair of neighboring qubits. Repeating T steps.
- Intuitively, the larger T, the more random the circuit becomes

Theorem 5

Fix $d = 2^n$, $\epsilon > 0$, $k \leq \sqrt{d}$, and let $\mathrm{G} \subset U(d^2)$ be a universal gate set containing inverses. Then, the set of all G-local random circuits of size T forms an ϵ-approximate k-design if

$$T \geq Cn[\log_2(k)]^2 k^{9.5} \left(nk + \log\left(\frac{1}{\epsilon}\right)\right),$$

where $C > 0$ is a (large) constant which depends on G

$n^2 k^{11}$

Local random circuits

Theorem 3: $2k$-design has $\left(\frac{d^2}{k}\right)^k C_\delta(U) \geq r+1$

Theorem 5: k-design if $\mathrm{T} \geq n^2 k^{11}$ size local circuit

- Theorem 3 + Theorem 5

Corollary 3 (Polynomial relation between circuit size and circuit complexity for local random circuits

Fix $\delta \in (0,1)$, $r \leq 2^{n/2}$ and set $T \geq Cn^2 \left(\frac{\log_2(n) r}{n}\right)^{11}$. Then, the set of all G-local circuits of size T contains at least $\tilde{C} 2^{\log(n) r}$ unitaries that obey $C_\delta(U) > r$. Here, $C, \tilde{C} > 0$ are constants that implicitly depend on δ and G.

$\therefore$ Strong complexity grows as $\Omega(T^{1/11})$

- Lower bound: for $\epsilon \leq 1/4$ and $k \leq d^{1/2}$, the size of random circuits on n qudits must be at least

$$T \geq \frac{2kn \log q}{q^4 \log k} \quad \text{to form an } \epsilon\text{-approximation } k\text{-design}$$

Relating two conjectures

- Fix $q = 2, d = 2^n$ (n qubits) and suppose that the aforementioned lower bound were not necessary, but also approximately sufficient:
  G-local circuits of size $T \simeq \frac{2nk}{\log_2(n)}$ generate sufficiently accurate approximat $2k$-designs
- Then, G-local circuits of size T contain at least $d^{2k}/(k!)^2$ elements with circuit complexity $r \simeq T$. If we assume that $T \leq \frac{2n}{\log_2(n)}\sqrt{d}$, then this bound can be simplified further as

$$K \gtrsim \frac{d^{2k}}{(k!)^2} = 2^{2nk-2\log(k!)} \gtrsim 2^{2k(n-\log(k))} = d^{\frac{k}{2}} = 2^{\frac{nk}{2}} \simeq 2^{\log_2(n)T} \geq 2^T$$

→ Conjecture 1!!

Conjecture 2 (Linear growth in complexity)
G-local circuits of size $T = O(nk)$ form approximate k-designs

Linear growth in design at large local dimension

Theorem 6

Random quantum circuits on n qudits of local dimension q form approximate unitary k-designs when the circuit size is $T = O(n^2 k)$ for some $q > q_0$, where q_0 depends on the size of the circuit.

Theorem 3 + Theorem 6

Corollary 4 (Linear complexity growth). *Given the set of local random circuits of size T at large q, most circuits have strong complexity $\Omega(T)$*, i.e. *growing linearly in T for a long time.*

Stochastic and Time-Dependent Models

Continuous-time analogue of random circuits

- Time-dependent Hamiltonian with random couplings
- Captures chaotic / Scrambling dynamics

Random all-to-all 2-body interactions & Gaussian random couplings

$$H_s = \sum_{i<j} \sum_{\alpha,\beta} J_{s,i,j,\alpha,\beta} S_i^\alpha S_j^\beta$$

$$U_t = \prod_{s-=1}^{t} e^{-iH_s \delta t}$$

where S_i^α is a Pauli operator acting on site i with $\alpha = \{0,1,2,3\}$, $J \sim N(0, \sigma^2)$

Theorem 7

Theorem 7 (Corollary 10 in [25]). *For $d = 2^n$ and $\epsilon > 0$, Then the ensemble of time-evolutions by stochastic Hamiltonians in Eq. (3.4), forms an ϵ-approximate k-design for times*

$$t \geq C\lceil \log_2(k) \rceil^2 k^{9.5} (nk + \log(1/\epsilon)), \tag{3.6}$$

where $C > 0$ is a constant.

$C \sim 1/J$

Nearly time-independent Hamiltonian dynamics

There is another random quantum circuit-like construction of a time-dependent Hamiltonian with varying couplings over discrete time steps. This "nearly time-independent" model of [26] forms k-designs in a depth $O(n^2 k)$ up to moments $k = o(\sqrt{n})$, achieving the nearly optimal lower bound with a linear growth in design for a short time.

Consider a 1d system of n qudits, with $d = q^n$, and define a time-dependent set of random couplings

$$\mathcal{J}(t,g) = \Big\{ \lambda/(\lfloor t/2 \rfloor + 1)\,, \quad \lambda \in [-g/2, g/2] \Big\}\,, \tag{3.7}$$

as well as two ensembles of Hamiltonians with time-dependent couplings

$$\mathcal{E}_Z(t) = \Big\{ -\sum_{j<k} h_{jk} Z_j Z_k - \sum_j b_j Z_j\,, \quad \text{with} \quad h_{jk} \in \mathcal{J}(t,h)\,, \; b_j \in \mathcal{J}(t,b) \Big\} \tag{3.8}$$

$$\mathcal{E}_X(t) = \Big\{ -\sum_{j<k} h_{jk} X_j X_k - \sum_j b_j X_j\,, \quad \text{with} \quad h_{jk} \in \mathcal{J}(t,h)\,, \; b_j \in \mathcal{J}(t,b) \Big\}\,, \tag{3.9}$$

where $g = \lfloor t/2 \rfloor/2$ and $b = \lfloor t/2 \rfloor + 1/2$. We then define the time-evolution of our system: we evolve by an X-type Hamiltonian $H_X \sim \mathcal{E}_X$ at even time steps and a Z-type Hamiltonian $H_Z \sim \mathcal{E}_Z$ at odd time steps. Then the unitary time-evolutions form an ϵ-approximate k-design for $k = o(n^{1/2})$, after T time steps, where

$$T \geq (k + 1/2 + (1/n)\log_2(1/\epsilon))\,. \tag{3.10}$$

This construction forms unitary k-designs almost linearly in time, with the caveat that the time scale is limited to $\sim \sqrt{n}$. Thus we get a linear growth in design at early times, but not exponentially in n. Consequently, this implies a linear growth in complexity at (very) early times.

Complexity growth and black hole interiors

In AdS/CFT, an eternal AdS-Schwarzschild black hole grows linearly in time ($t \sim e^{n}$)

Conjecture (Complexity = Volume):

- Quantum complexity of the dual CFT state is the long-time linearly increasing **quantity** which captures the wormhole growth.
- Complexity of the boundary TFD state equals the spatial volume behind the horizon

Limits of the k-Design Approach for Holography

The connection between unitary designs and quantum complexity will likely not inform complexity growth in holography as evolution by time-independent Hamiltonian will not converge to approximate designs.

→ we need to explore properties beyond the Haar-randomness of the evolution to study complexity growth in holography

Strong complexity in the bulk

Claim: Strong definition of complexity is congruent with expectations form the bulk and might be more suited for holography than the standard definition in terms of the circuit complexity.

$$O(t) = e^{-iHt} O e^{iHt}$$

Switchback effect (complexity growth in holography)

	(Traditional) Circuit Complexity	Proposed Strong Complexity
Definition	Minimal gates to build a unitary	Minimal measurement complexity to distinguish from maximally mixed state
Switchback	Gate cancellations outside the operator's lightcone delay growth	Simple measurements outside the lightcone can still distinguish
After scrambling	Operators spread over all qubits → gates no longer cancel → linear growth	Distinguishing requires global measurements → linear growth
Interpretation	Circuit cancellation	Information access-delay

Strong complexity in the bulk

A more interesting example, where the strong and weak definitions of complexity differ, is that of one-clean qubit. This is essentially the argument given in Lemma 2, to prove that measurement complexity is a stronger definition than standard circuit complexity. Consider a simple initial state $|\psi_0\rangle$, which has been evolved for an exponential time such that $|\psi(t)\rangle$ is maximally complex. If we add a single unentangled qubit to the state $|\psi(t)\rangle \otimes |0\rangle$, then the minimal circuit complexity will be unchanged, but maximal potential complexity increases and the complexity of the state can continue to grow for a long time until it saturates at the new maximal value. For the complexity of a distinguishing measurement, adding a single clean qubit resets the complexity to an order one value, as the optimal measurement is simply the projection onto the clean qubit. Ref. [7] proposed the notion of uncomplexity as the difference of the complexity of a state or unitary from its maximal complexity, where uncomplexity can be thought of as a resource to do useful computation. As we described, our strong definition of complexity directly encodes this potential for useful quantum computation.

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Most states have high complexity

Random pure states $|\psi\rangle\langle\psi|$ behave like the maximally mixed state ρ_0 in expectation.

$$\mathbb{E}_{|\psi\rangle}\left[\mathrm{Tr}\left(M|\psi\rangle\langle\psi|\right)\right] = \mathrm{Tr}\left(M\mathbb{E}_{|\psi\rangle}\left[|\psi\rangle\langle\psi|\right]\right) = \mathrm{Tr}\left(M\rho_0\right)$$

Concentration of measure ensures that deviations from this average case behavior are exponentially suppressed in concrete instances

$$\Pr\left[|\mathrm{Tr}\left(M(|\psi\rangle\langle\psi| - \rho_0)\right)| \geq \tau\right] \leq 2\exp\left(-\frac{d\tau^2}{9\pi^3}\right) \quad \text{for any} \quad \tau \geq 0$$

We refer to Proposition 1 in the appendix for a proof of this well-known result. We can combine this assertion with a union bound (Boole's inequality) to conclude for any $r \in \mathbb{N}$ and $\delta \in (0,1)$

$$\begin{aligned}\Pr\left[\mathcal{C}_\delta(|\psi\rangle) \leq r\right] &= \Pr\left[\max_{M \in \mathsf{M}_r} |\mathrm{Tr}\left(M(|\psi\rangle\langle\psi| - \rho_0)\right)| \geq 1 - d^{-1} - \delta\right] \\ &\leq 2.0072|\mathsf{M}_r|\exp\left(-\frac{d(1-\delta)^2}{9\pi^3}\right). \end{aligned} \tag{5.3}$$

Suppose that M_r arises from combining at most r elements of a fixed universal gate set $\mathsf{G} \subset U(q^2)$. A naive counting argument reveals $|\mathsf{M}_r| \leq 2dn^r|\mathsf{G}|^r$. We conclude that the $\Pr\left[\mathcal{C}_\delta(|\psi\rangle) \leq r\right]$ remains exponentially suppressed (in $d = q^n$) until

$$r \simeq \frac{q^n}{\log(n)}. \tag{5.4}$$

To summarize, a random state is exceedingly likely to have an exponentially large strong δ-state complexity.

Most high-complexity states far apart

We show this statement by induction based on two features of Haar random states. Firstly, we use the main result from the previous subsection. Choose $r \lesssim q^n / \log(n)$ such that Eq. (5.3) ensures

$$\Pr\left[\mathcal{C}_\delta(|\psi\rangle) \leq r\right] \leq 2.0072|\mathrm{M}_r| \exp\left(-\frac{d(1-\delta)^2}{9\pi^3}\right) \leq \frac{1}{2}. \tag{5.5}$$

The parameter r is chosen such that Haar random states exceed this complexity with probability 1/2. Concentration of measure also implies that a Haar-random state is very likely to be far away from any fixed state $|\phi\rangle\langle\phi|$. For any $\Delta \in (0,1)$,

$$\boxed{\Pr\left[\tfrac{1}{2}\left\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \right\|_1 \leq 1-\Delta\right] = \Pr\left[|\langle\psi|\phi\rangle|^2 \geq \Delta^2\right] \leq 3\exp\left(-\frac{\Delta^2 d}{9\pi^3}\right). \qquad (5.6)}$$

This bound readily follows from Eq. (5.2) (set $M = |\phi\rangle\langle\phi|$) and elementary modifications.

Most high-complexity states far apart

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