Instance-Optimal Quantum State Certification with Entangled Measurements

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Joint work with Ryan O'Donnell (CMU)

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Outline

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Quantum State Certification

- A tester is given the **complete description** of a quantum state $\sigma \in \mathbb{C}^{d \times d}$ and n **copies** of an unknown state $\rho \in \mathbb{C}^{d \times d}$.
- Promised that $\rho = \sigma$ or $\|\rho - \sigma\|_1 \geq \epsilon$.
- ϵ-**certifying** σ: distinguishing between these cases with high probability.

Motivation

- Benchmarking quantum devices.
- Verifying heuristic quantum learning algorithms.

Also, a direct quantum analogue of **identity testing** in classical distribution testing:

Given the description of a distribution q, and n samples from an unknown distribution p, test whether $p = q$ or $\|p - q\|_1 \geq \epsilon$.

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- [BOW19] developed an algorithm using $\mathcal{O}(d/\epsilon^2)$ copies to certify any state.
- This establishes a tight **worst-case** complexity: $\Theta(d/\epsilon^2)$.

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- But the problem could be much easier for other choices of σ!

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- This establishes a tight **worst-case** complexity: $\Theta(d/\epsilon^2)$.
- But the problem could be much easier for other choices of σ!
- For e.g., when σ is pure, $\mathcal{O}(1/\epsilon^2)$ copies suffice [MdW16].

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[MdW16]: A survey of quantum property testing; Montanaro-de Wolf 2016.

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[CLO22]: Toward instance-optimal state certification with incoherent measurements; Chen-Li-O'Donnell 2022
[CLHL22]: Tight bounds for quantum state certification with incoherent measurements; Chen-Li-Huang-Liu 2022

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- We even have classical bounds for instance-optimal identity testing in various forms [**VV17, DK16, BCG19**].

[VV17]: An automatic inequality prover and instance optimal identity testing; Valiant-Valiant 2017.
[DK16]: A new approach for testing properties of discrete distributions; Diakonikolas-Kane 2016.
[BCG19]: Distribution testing lower bounds via reductions from communication complexity; Blais-Canonne-Gur 2019.

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- We even have classical bounds for instance-optimal identity testing in various forms [**VV17, DK16, BCG19**].
- However, when quantum testers are unrestricted, instance-optimal bounds were not known.

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Nearly Instance-Optimal Bounds

Theorem (Main Result)

With fully entangled measurements, the copy complexity n of ϵ-certifying σ satisfies

$$\tilde{\Omega}\left(\frac{d \cdot F(\underline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right) \leq n \leq \tilde{\mathcal{O}}\left(\frac{d \cdot F(\overline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right).$$

$\underline{\sigma}, \overline{\sigma}$ are variants of σ constructed by zeroing out suitable eigenvalues adding up to $\mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2)$ respectively.

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- For $\sigma = \mathbb{1}/d$, $n = \tilde{\Theta}(d/\epsilon^2)$.
- For pure σ, $n = \tilde{\Theta}(1/\epsilon^2)$.

Mixedness Testing Lower Bound

- Our main theorem recovers the mixedness testing bounds up to $\log(d/\epsilon)$ factors.
- Directly applying our techniques to mixedness testing, we actually recover $\Omega(d/\epsilon^2)$ without any log factors!
- With our new lower bound technique, this proof is much simpler than that of [**OW15**]!

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- $\mathrm{d}_{\mathrm{tr}}(\sigma, \mathbb{1}/d)$ depends only on σ's eigenvalues.
- For such **spectrum tests**, weak Schur sampling is known to be optimal [**CHW07,MdW16**].
- [**OW15**] then prove the mixedness testing lower bound by analyzing the resulting Schur-Weyl distributions.

[**CHW07**]: Weak Fourier-Schur sampling, the hidden subgroup problem, and the quantum collision problem; Childs-Harrow-Wocjan 2007.

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We need a new way to prove the mixedness testing lower bound!

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- Relate to χ^2-divergence: $\mathrm{d}_{\mathrm{TV}} \leq \frac{1}{2}\sqrt{\mathrm{d}_{\chi^2}}$.
- n must be large enough so that $\mathrm{d}_{\chi^2}(\mathbb{E}_\theta[q_\theta^{\otimes n}] \| q^{\otimes n}) \geq c$.
- Use the *Ingster-Suslina method* [IS12] to explicitly compute d_{χ^2} and easily upper bound it.

[IS12]: Nonparametric goodness-of-fit testing under Gaussian models; Ingster-Suslina 2012.

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- Given n copies of an unknown state, distinguish between the following equally likely cases:
  1. We receive n copies of a fixed state σ.
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- **New Tool:** A quantum Ingster-Suslina method to explicitly compute D_{χ^2} and easily upper bound it!

Quantum χ^2-divergence

Given two states ρ, σ, let $\Delta = \rho - \sigma$. Then,

$$\mathrm{D}_{\chi^2}(\rho\|\sigma) = \mathrm{tr}\left(\sigma^{-1}\Delta^2\right) = \mathrm{tr}\left(\sigma^{-1}\rho^2\right) - 1.$$

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Want to upper bound $\mathrm{D}_{\chi^2}(\mathbb{E}_\theta[\sigma_{\boldsymbol{\theta}}^{\otimes n}]\|\sigma^{\otimes n})$.

A Quantum Ingster-Suslina Method

We show:

$$\mathrm{D}_{\chi^2}(\mathbb{E}_\theta[\sigma_{\boldsymbol{\theta}}^{\otimes n}]\|\sigma^{\otimes n}) + 1 = \mathbb{E}_{\boldsymbol{\theta},\boldsymbol{\theta'}}(1 + Z(\boldsymbol{\theta},\boldsymbol{\theta'}))^n$$

where

$$Z(\boldsymbol{\theta},\boldsymbol{\theta'}) = \mathrm{tr}\left(\sigma^{-1}\Delta_{\boldsymbol{\theta}}\Delta_{\boldsymbol{\theta'}}\right) \quad \text{and} \quad \Delta_{\boldsymbol{\theta}} = \sigma_{\boldsymbol{\theta}} - \sigma.$$

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Usage for certification lower bounds:

1. Construct a suitable mixture of alternatives $\{\sigma_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}}$.
2. Upper bound $\mathbb{E}_{\boldsymbol{\theta},\boldsymbol{\theta}'} \exp\big(nZ(\boldsymbol{\theta},\boldsymbol{\theta}')\big)$.

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  This is $\Omega(1)$ only if $n = \Omega(d/\epsilon^2)$.

Lower Bounds for Nearly Mixed States

The same technique also works for nearly maximally mixed states! In general, for well-conditioned states σ, we show:

$$n \geq \Omega\left(\frac{d^{5/2}}{\epsilon^2 \cdot \|\sigma^{-1}\|_2}\right).$$

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But this bound is not always strong enough!

Consider, e.g., σ with spectrum $= (\Omega(1/d), \ldots, \Omega(1/d), 1/d^2)$.

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$\Rightarrow$ We place too much emphasis on the smallest eigenvalues!

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$$\begin{pmatrix} 1/7 & & & & & & \\ & 1/8 & & & & & \\ & & \ddots & & & & \\ & & & 1/16 & & & \\ & & & & 1/17 & & \\ & & & & & 1/32 & \\ & & & & & & \epsilon/2 \\ & & & & & & & \epsilon/4 \end{pmatrix}$$

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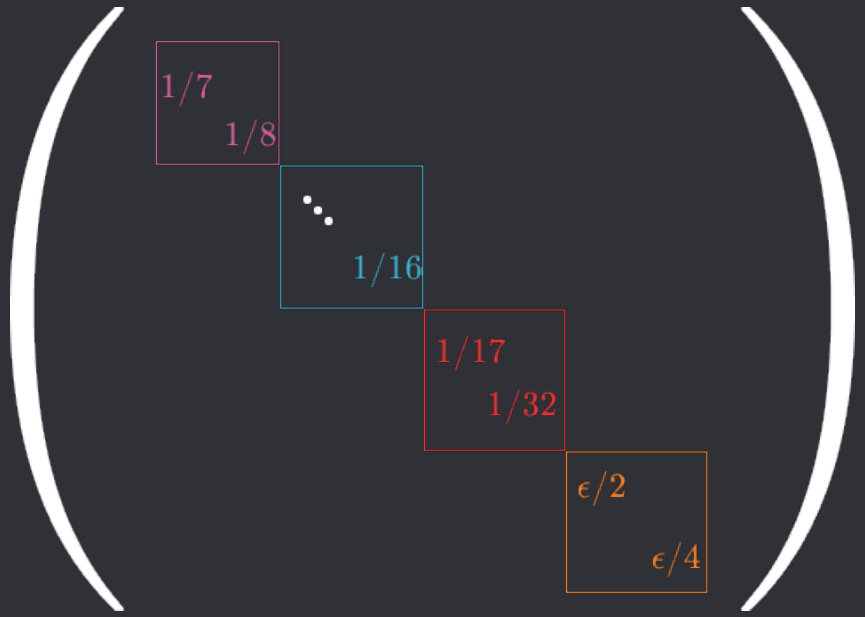

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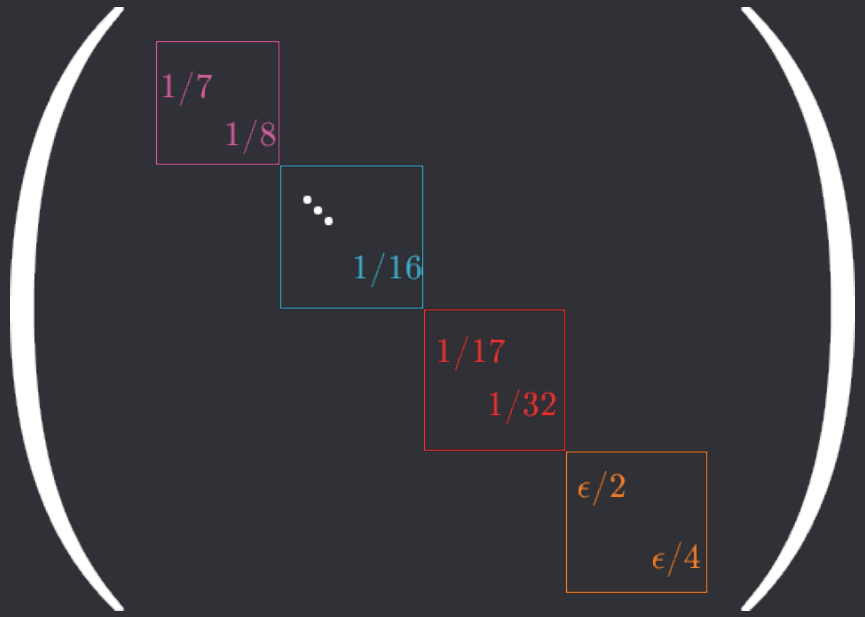

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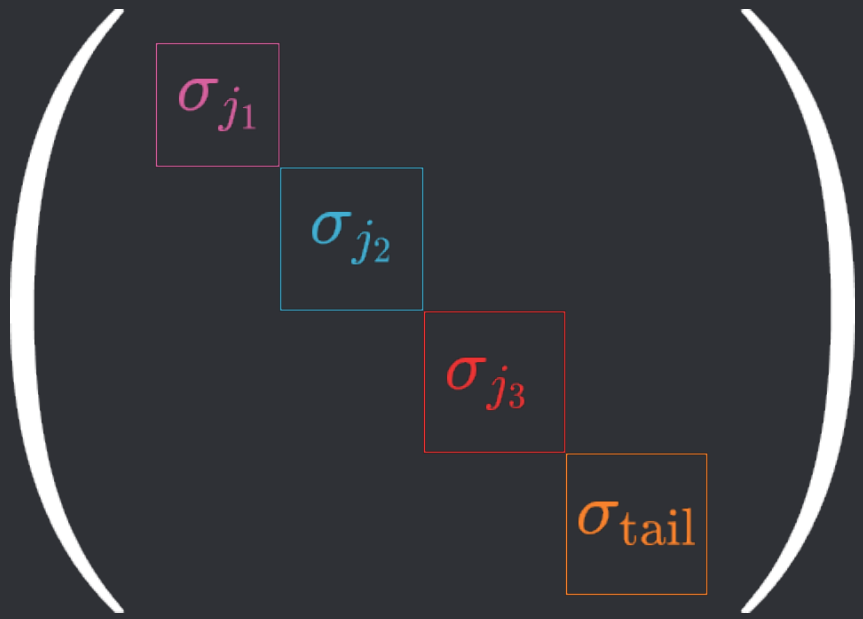

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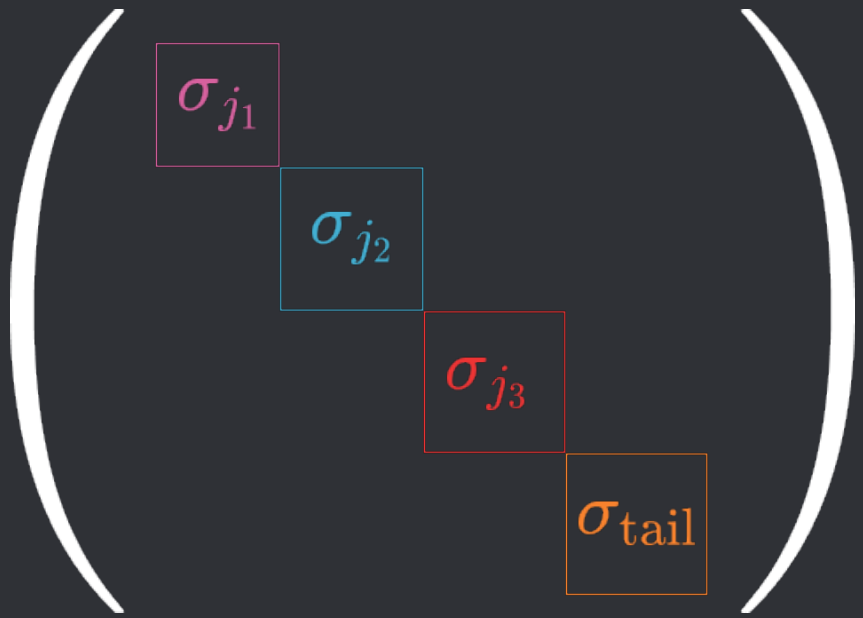

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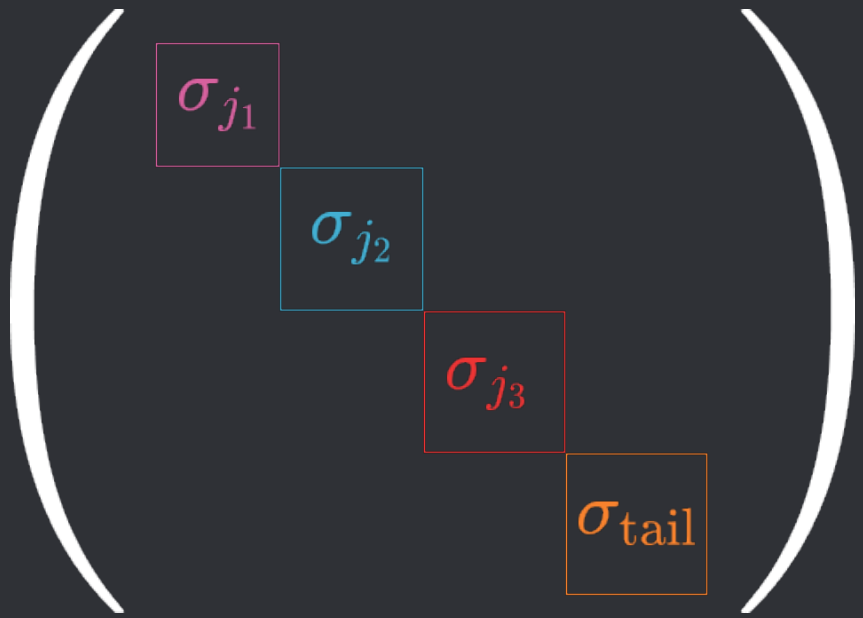

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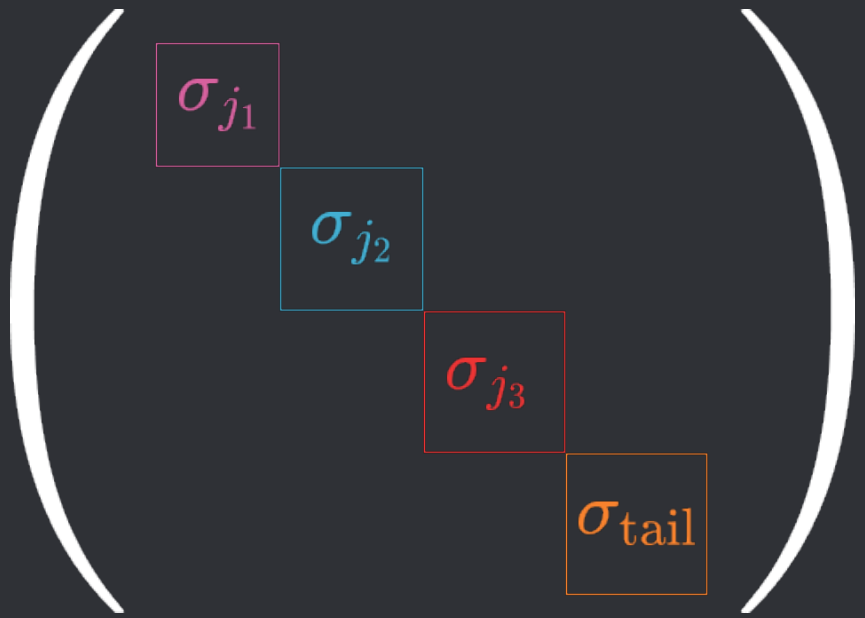

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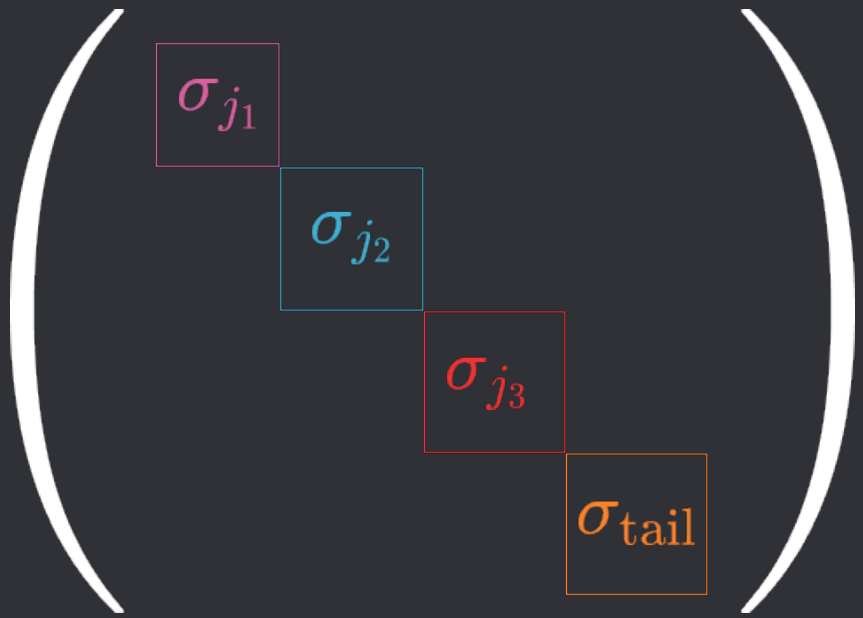
- Let $\vec{\boldsymbol{U}} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m)$.



Mixture of Alternatives

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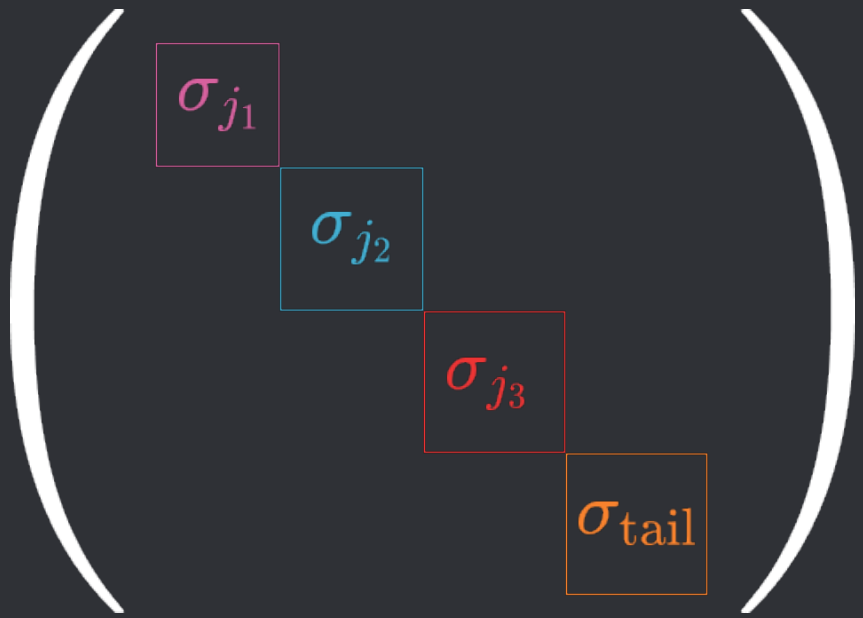

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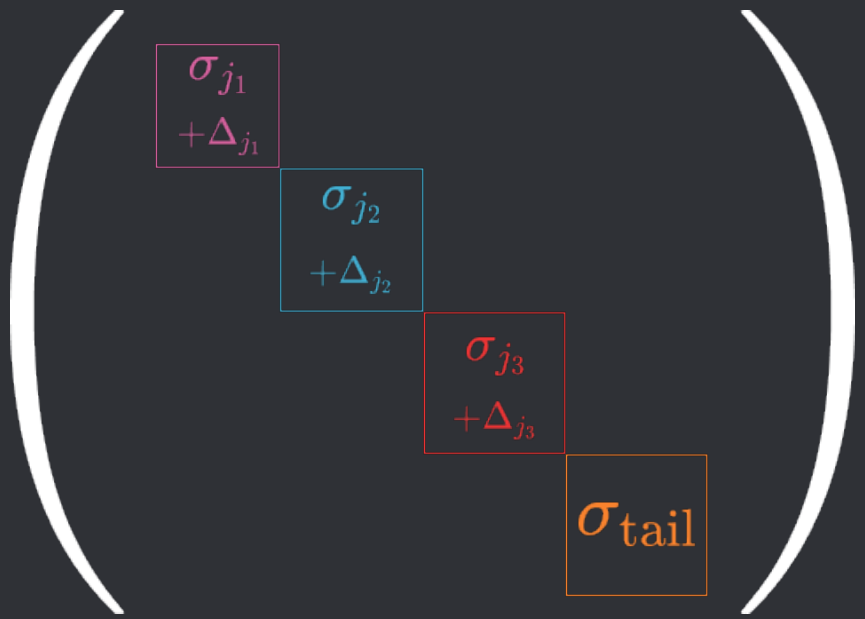

Decomposition into Independent RVs

$$\mathrm{D}_{\chi^2}(\mathbb{E}_{\vec{\boldsymbol{U}}}[\sigma_{\vec{\boldsymbol{U}}}^{\otimes n}]\|\sigma^{\otimes n}) = \mathbb{E}_{\vec{\boldsymbol{U}},\vec{\boldsymbol{V}}}\left[(1+Z(\vec{\boldsymbol{U}},\vec{\boldsymbol{V}}))^n\right] - 1,$$

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Decomposition into Independent RVs

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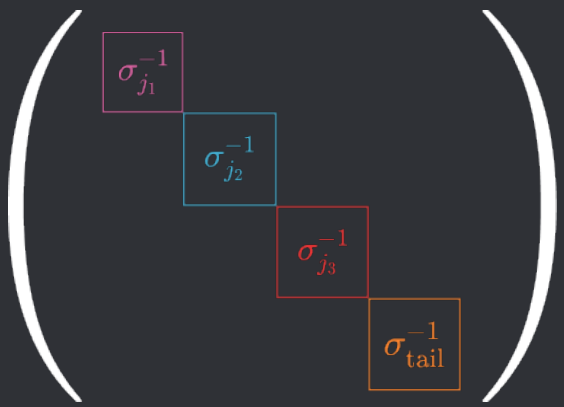

Decomposition into Independent RVs

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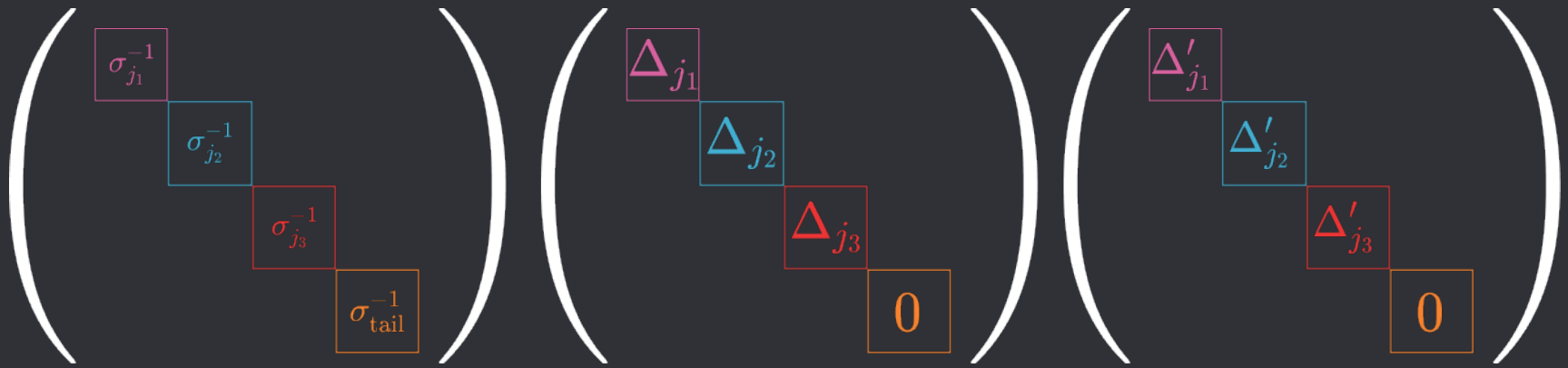

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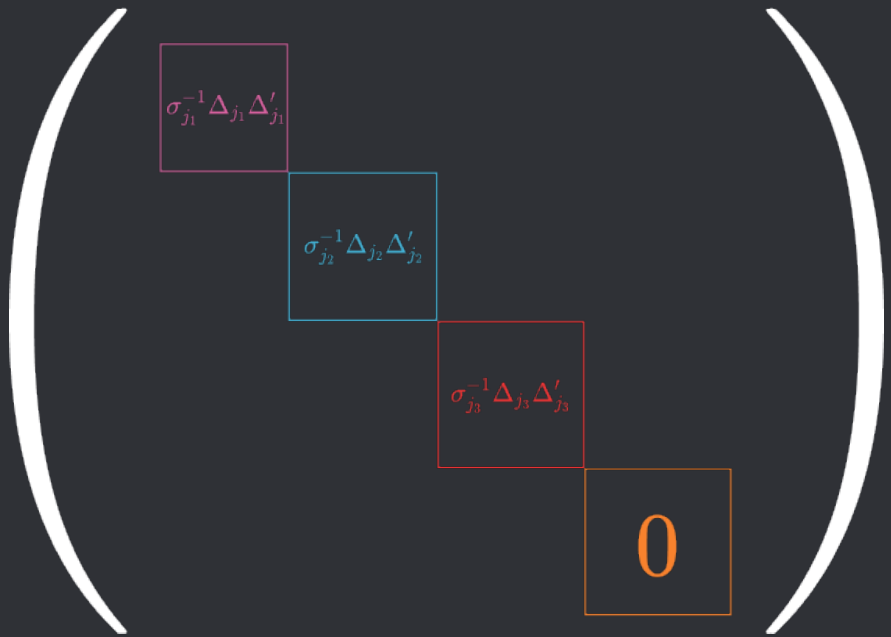

Decomposition into Independent RVs

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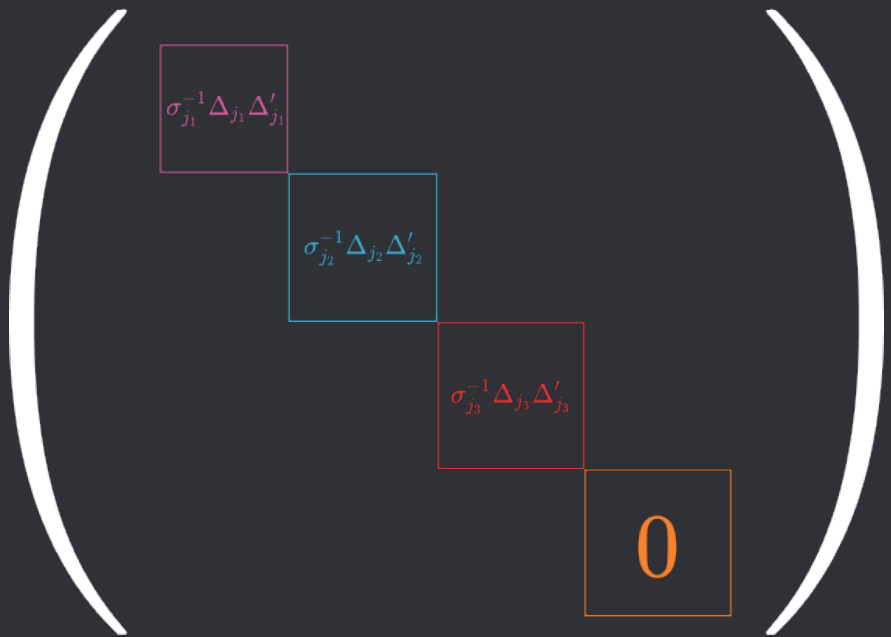

Final Steps

$$
\begin{aligned}
\mathrm{D}_{\chi^2} + 1 &\leq \mathbb{E}_{\vec{\boldsymbol{U}},\vec{\boldsymbol{V}}} \exp\Big(n \cdot Z(\vec{\boldsymbol{U}},\vec{\boldsymbol{V}})\Big) \\
&= \mathbb{E}_{\vec{\boldsymbol{U}},\vec{\boldsymbol{V}}} \exp\left(\sum_j n \cdot \mathrm{tr}\Big(\sigma_j^{-1}\Delta_j\Delta_j'\Big)\right) \\
&= \prod_j \mathbb{E}_{\boldsymbol{U}_j,\boldsymbol{V}_j} \exp\left(n \cdot \mathrm{tr}\Big(\sigma_j^{-1}\Delta_j\Delta_j'\Big)\right).
\end{aligned}
$$

Final Steps

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- Pick $\{\epsilon_j\}$.
- Corner cases: all buckets have $d_j = 1$ or $\|\rho\|_\infty \geq \frac{1}{2}$.

Details in the paper (Section 5)

Outline

Tools from Prior Work

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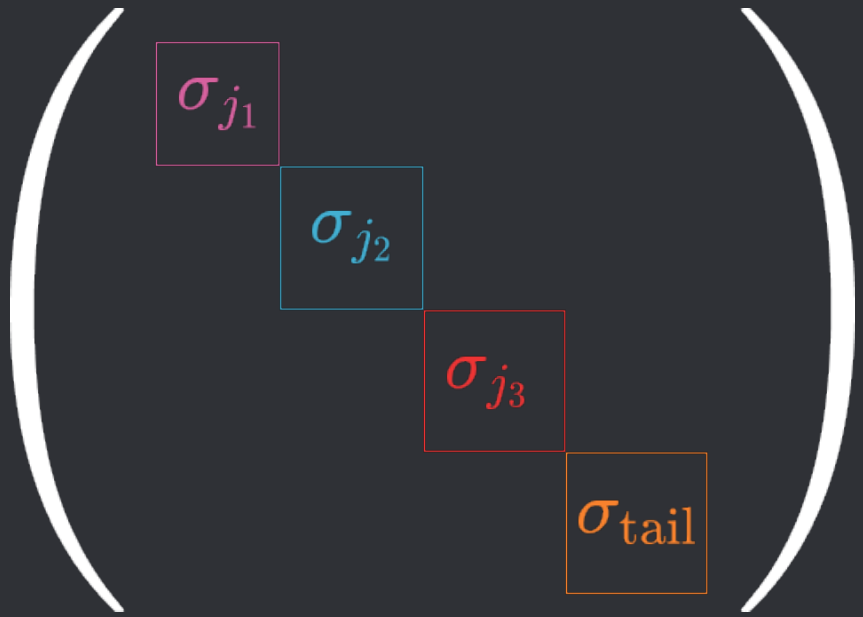
$\Rightarrow$ We replace this with an entangled-measurement certifier.

Theorem (Hilbert-Schmidt Tester from [BOW19])

There exists an algorithm HSCertify *that can distinguish between* $\rho = \sigma$ *and* $\|\rho - \sigma\|_2 \geq \epsilon$ *using* $\mathcal{O}(1/\epsilon^2)$ *copies of* ρ.

Upper Bound



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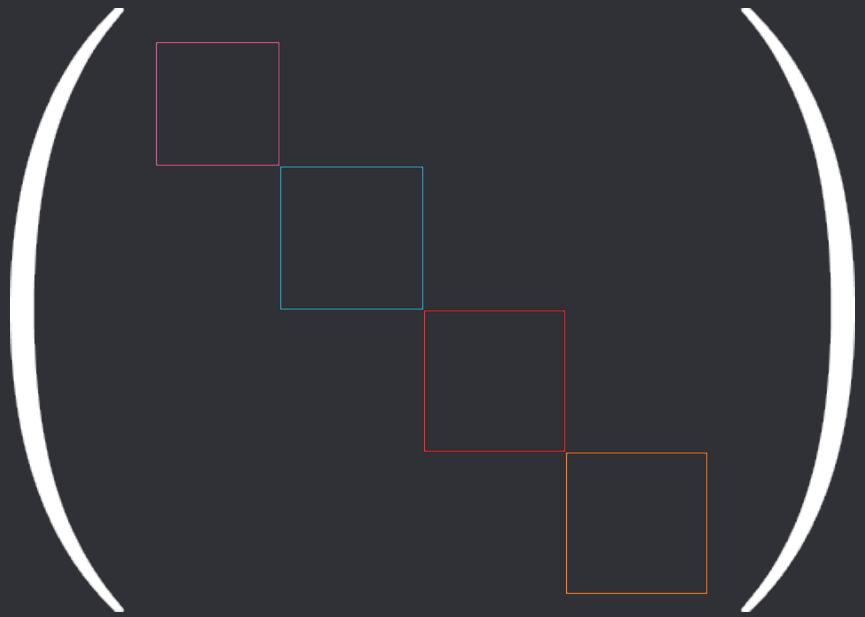
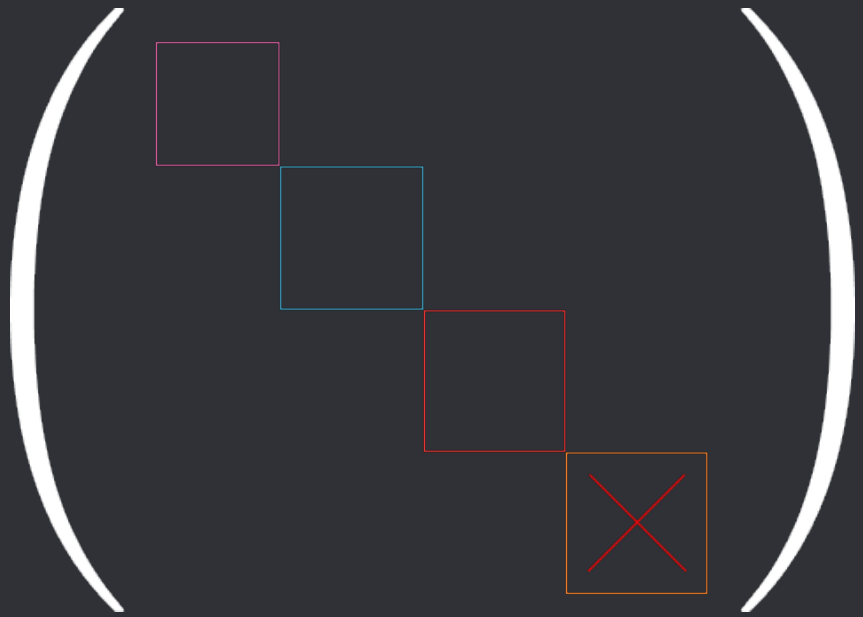


Upper Bound: Case 1

Case 1: The unknown state has too much weight on the tail.



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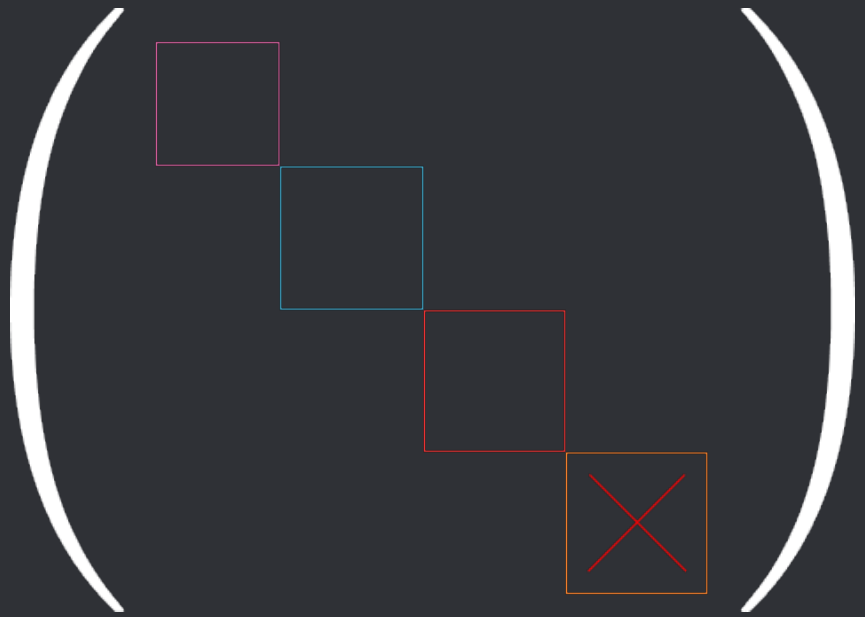
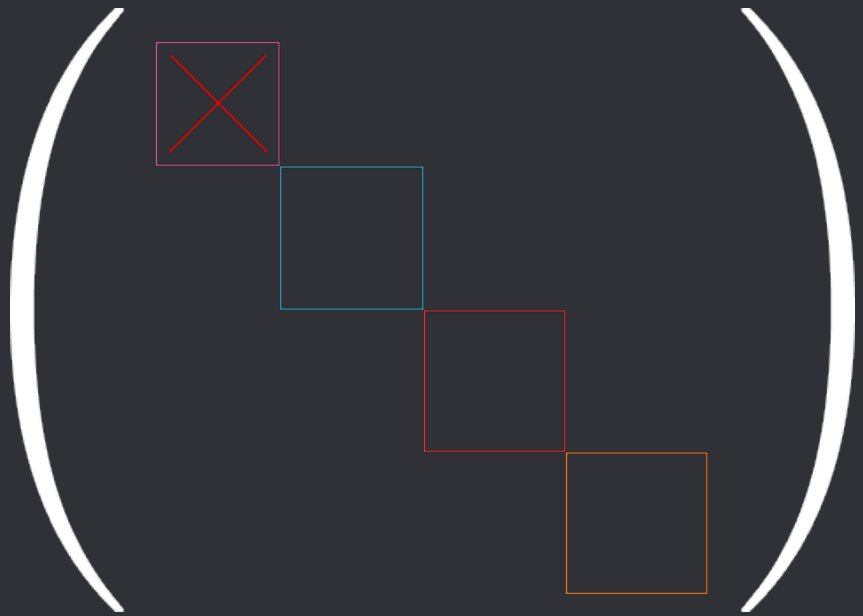
Can be tested for with $\mathcal{O}(1/\epsilon^2)$ (unentangled!) measurements (from [**CLO22**]).



Upper Bound: Case 2

Case 2: For some bucket j, $\|\rho_j - \sigma_j\|_1$ is too large.



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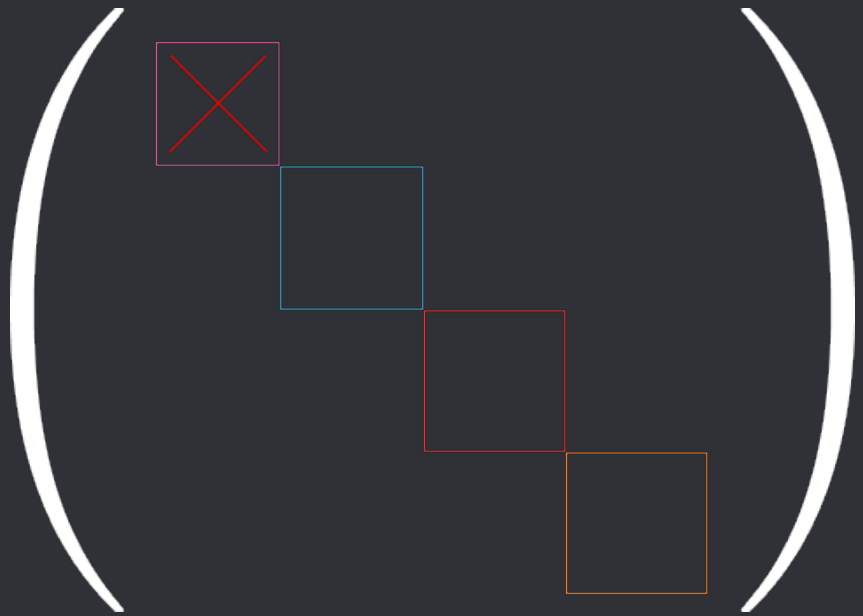
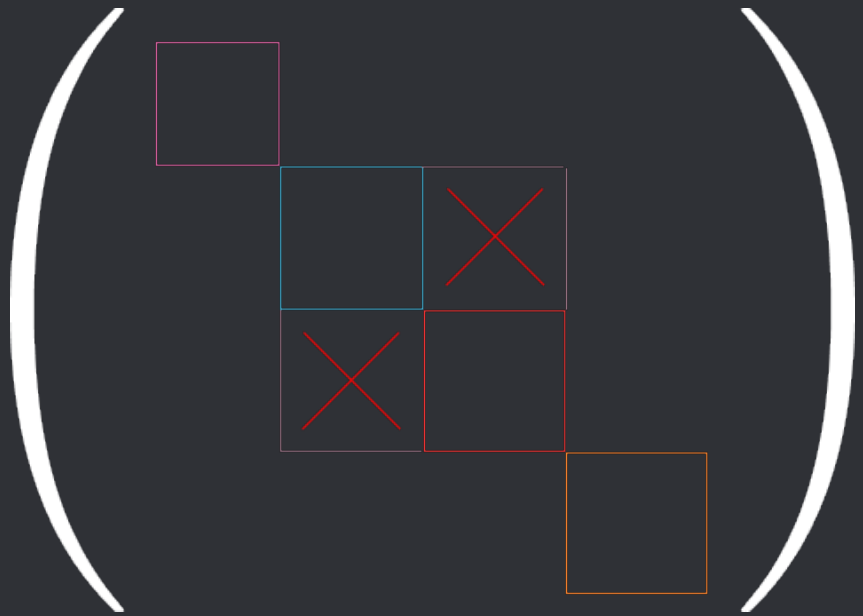
After simple pre-processing, projecting and then passing to HSCertify, this can be handled with $\tilde{\mathcal{O}}\left(\frac{d \cdot F(\overline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right)$ copies of ρ.



Upper Bound: Case 3

Case 3: For some buckets j, j', the non-principal submatrices are too far.



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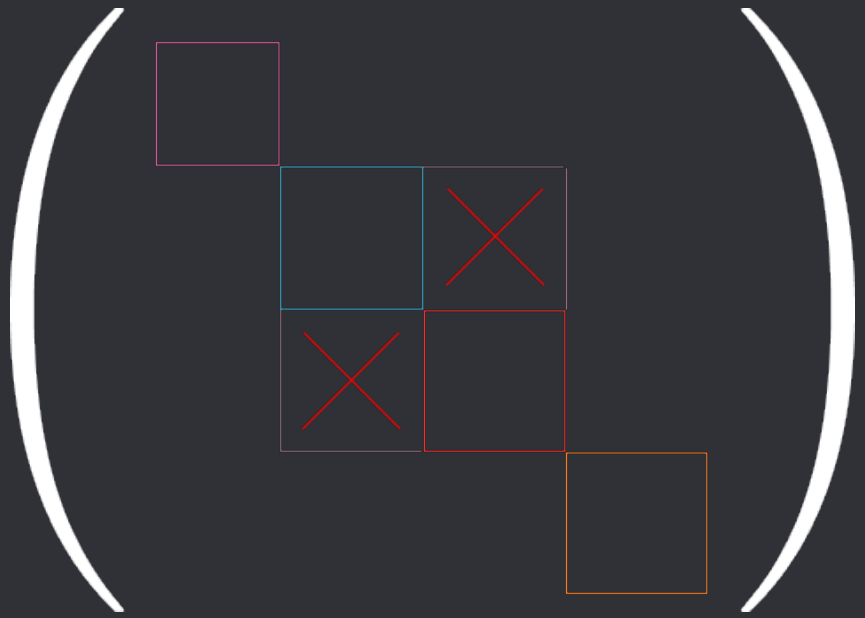

Assuming Case 2 does not hold, this can again be handled with $\tilde{\mathcal{O}}\left(\frac{d \cdot F(\overline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right)$ copies of ρ.

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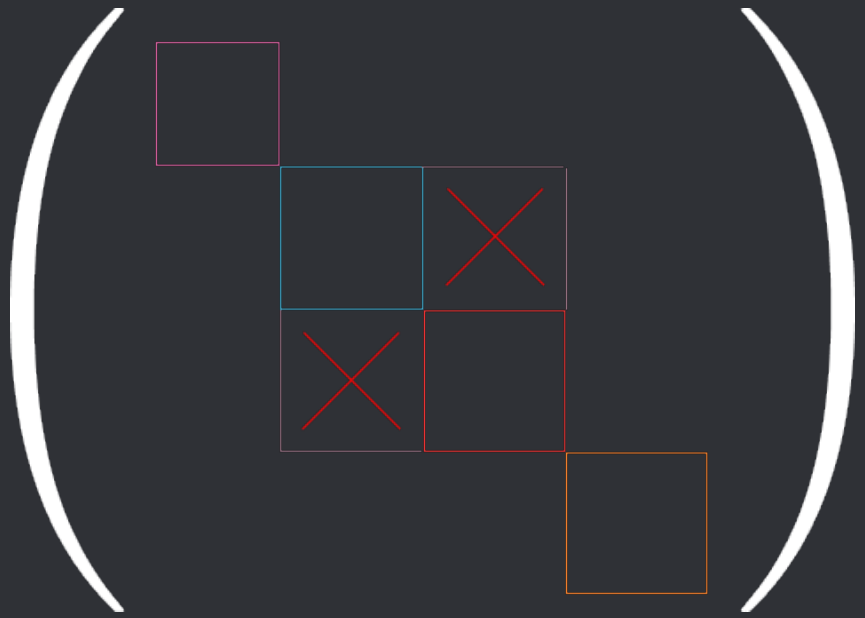

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Total complexity: Case 1 + Case 2 + Case 3 $= \tilde{\mathcal{O}}\left(\frac{d \cdot F(\overline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right)$

Discussion

We have shown nearly instance-optimal bounds for state certification with entangled measurements:

$$\tilde{\Omega}\left(\frac{d \cdot F(\underline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right) \leq n \leq \tilde{\mathcal{O}}\left(\frac{d \cdot F(\overline{\sigma}, \mathbb{1}/d)}{\epsilon^2}\right).$$

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- The amounts of mass removed to get $\underline{\sigma}, \overline{\sigma}$ do not match; can this be overcome?
- What about state certification with t-copy measurements, with $1 < t \ll d/\epsilon^2$? [CCHL21] have some partial results, but even worst-case bounds remain open.

[CCHL21]: A hierarchy for replica quantum advantage; Chen-Cotler-Huang-Li 2021.

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Thank you!

arxiv:2507.06010

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